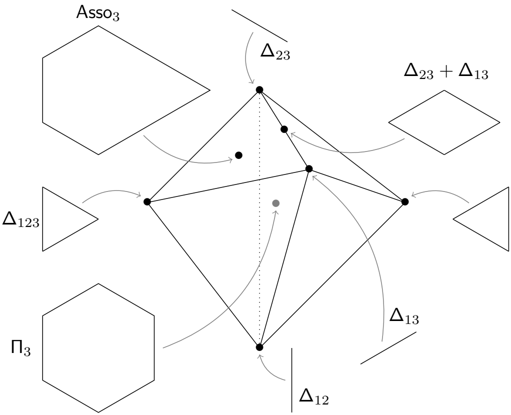
Geometric combinatorics of paths and deformations of convex polytopes

Germain Poullot



19 October 2023

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1. What is “Combinatorics of Polytopes”?

2. Generalized permutahedra
   - Deformations
   - Submodular Cone
   - Ongoing work

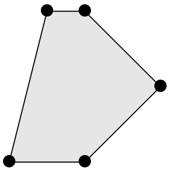
3. Max-slope Pivot Polytopes
   - Max-slope pivot rule
   - Poset of slopes
   - Pivot rule polytope of products of simplices

What is "Combinatorics of Polytopes"?

Polytopes

Definition

Polytope: convex hull of finitely many points in $\mathbb{R}^n$



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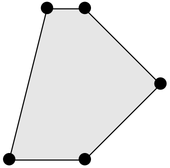
Polytope: convex hull of finitely many points in $\mathbb{R}^n$
bounded intersection of finitly many half-spaces in $\mathbb{R}^n$



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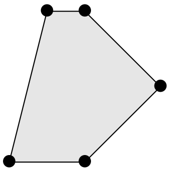

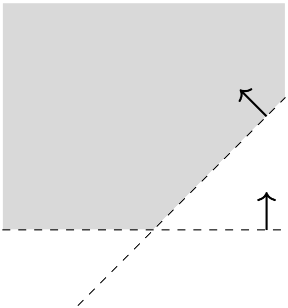

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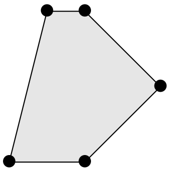

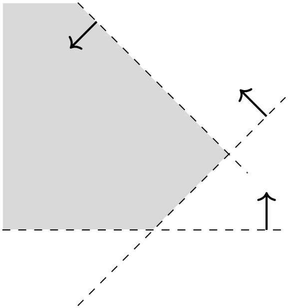

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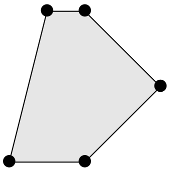

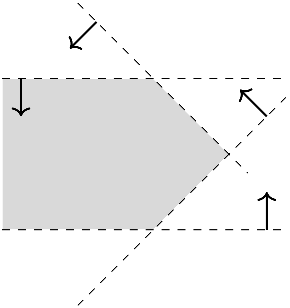

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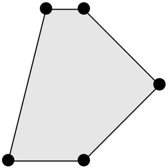
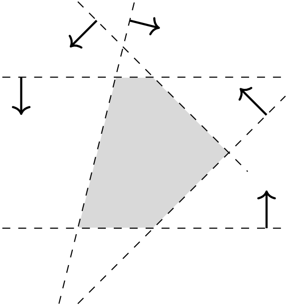




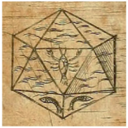
Representing polytopes

Tetrahedron
Fire

Hexahedron
Earth

Octahedron
Air

Dodecahedron
the Universe



Icosahedron
Water

Representing polytopes

Tetrahedron
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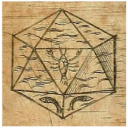

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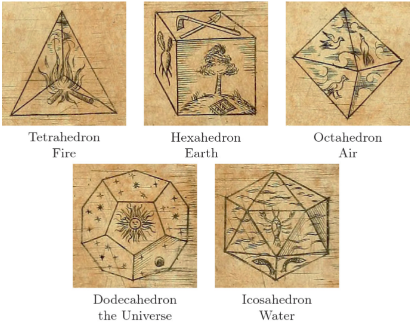
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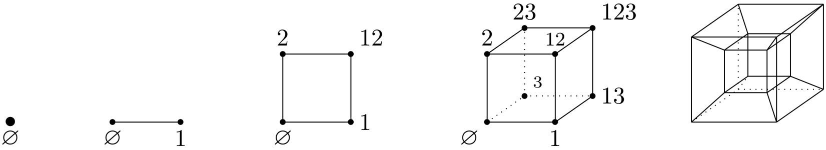
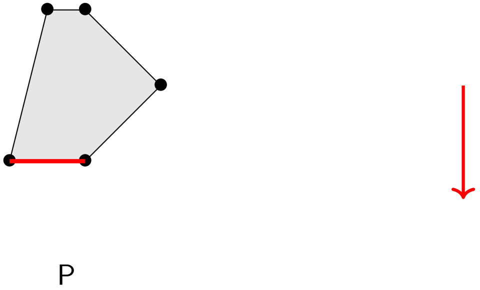

∅
∅
1
2
12
∅
1
23
123
2
12
3
13
∅
1

Definition

Face: $\mathrm{P}^{\boldsymbol{c}} := \left\{ \boldsymbol{x} \in \mathbb{R}^n \ ; \ \langle \boldsymbol{x}, \boldsymbol{c} \rangle = \max_{\boldsymbol{y} \in \mathrm{P}} \langle \boldsymbol{y}, \boldsymbol{c} \rangle \right\}$



Faces

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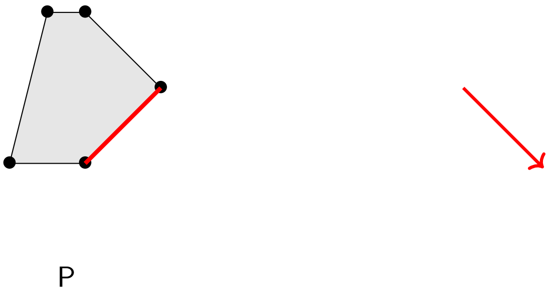

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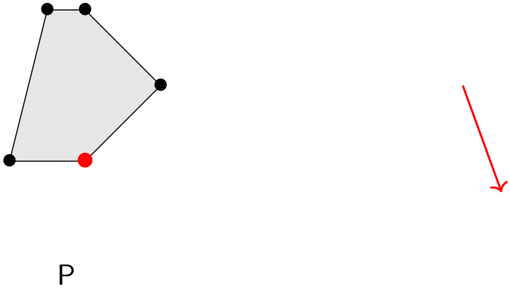
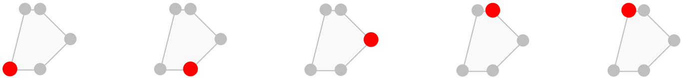


Face lattice

Face lattice: poset of inclusions of faces



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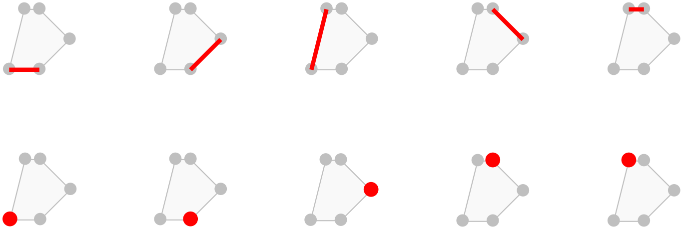

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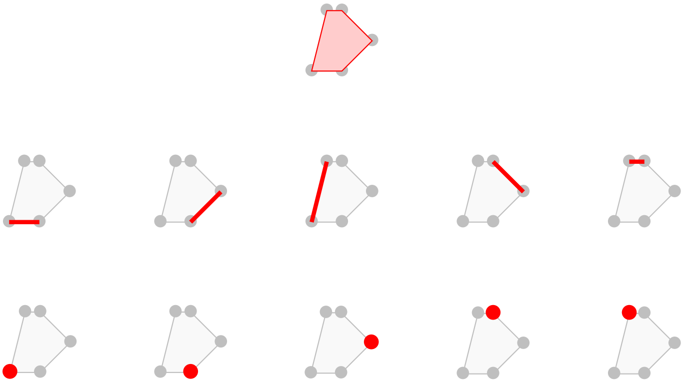

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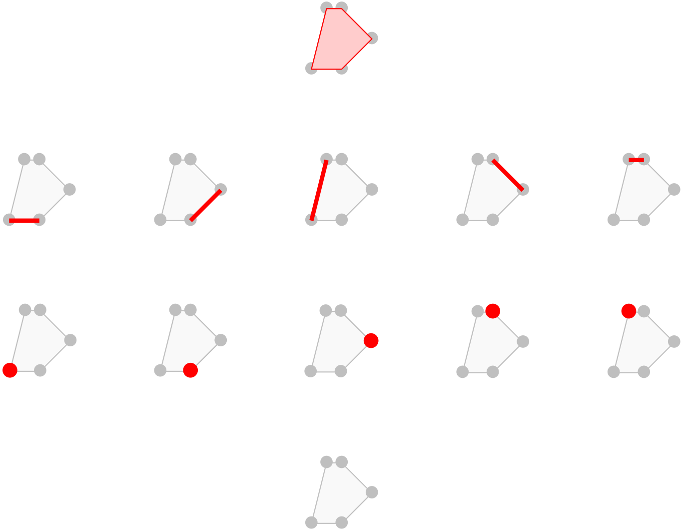

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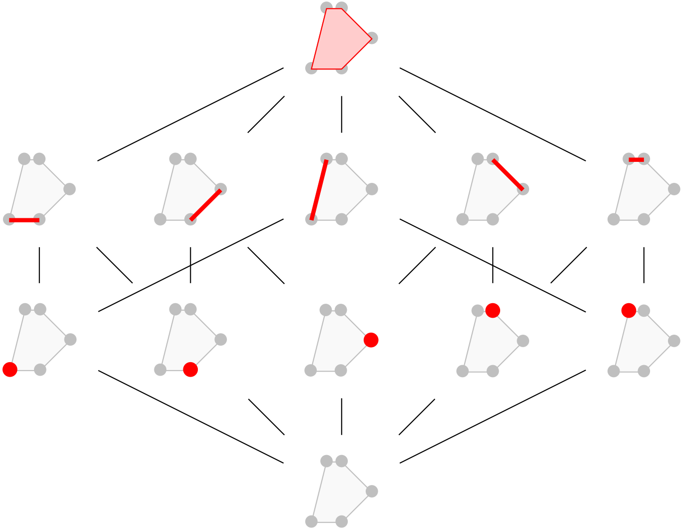
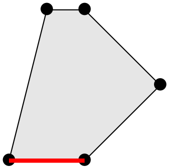
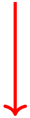


Normal fan

Definition

Normal cone of a face F: $\mathcal{N}_{\mathsf{P}}(\mathsf{F}) := \{\boldsymbol{c}\ ;\ \mathsf{P}^{\boldsymbol{c}} = \mathsf{F}\}$
Normal fan $\mathcal{N}_{\mathsf{P}}$: collection of $\mathcal{N}_{\mathsf{P}}(\mathsf{F})$ for F face of P





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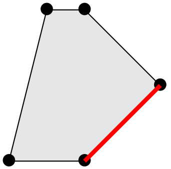

P

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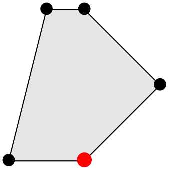
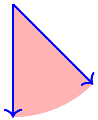


P



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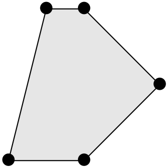

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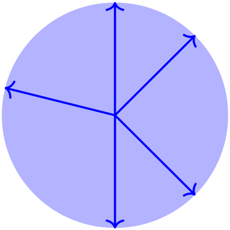

$\mathcal{N}_{\mathrm{P}}$

One way: Take a polytope $\rightarrow$ combinatorial info (e.g. face lattice)

Combinatorics of Polytopes

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Example (Permutahedron)

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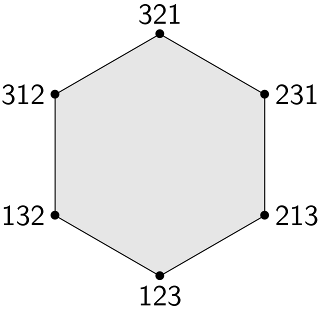

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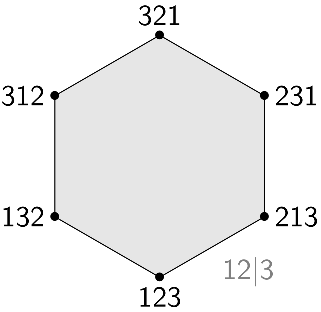

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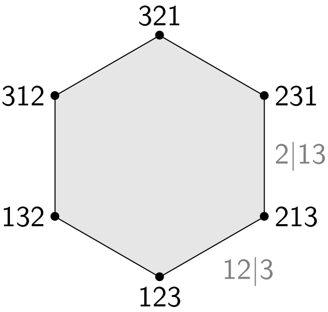

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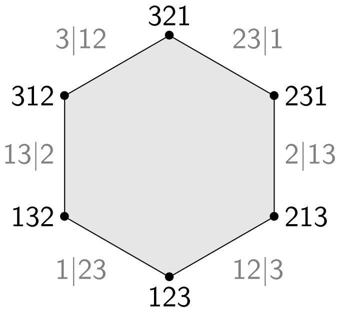

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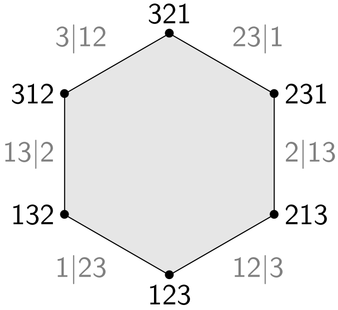
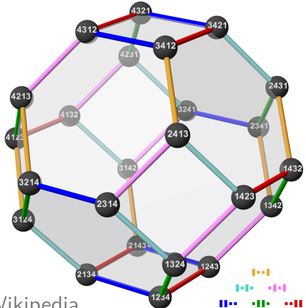




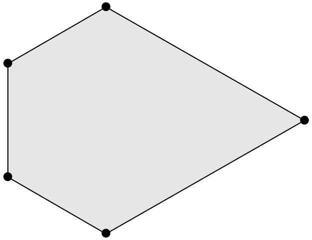
Wikipedia

Generalized permutahedra

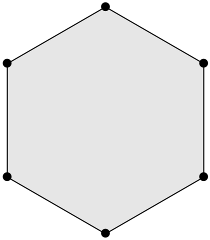
Coarsening: Choose maximal cones and merge them

Definition

Q is a *deformation* of P iff $\mathcal{N}_{\mathrm{Q}}$ coarsens $\mathcal{N}_{\mathrm{P}}$.



deformation of



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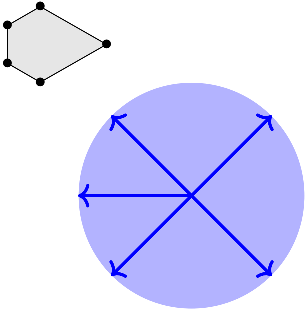

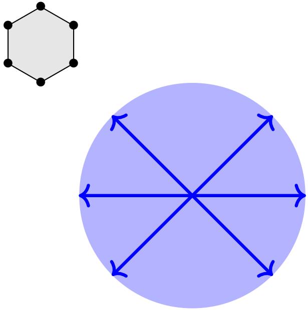

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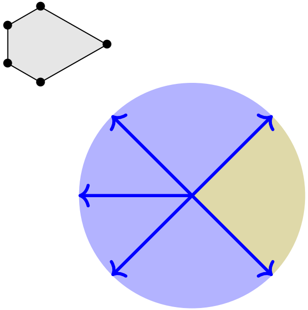
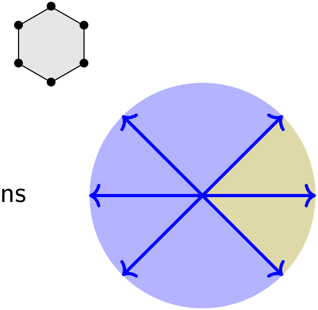


coarsens



Braid fan

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Braid fan: arrangement of hyperplanes $H_{i,j} := \{\boldsymbol{x} \ ; \ x_i = x_j\}$

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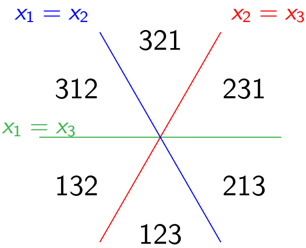
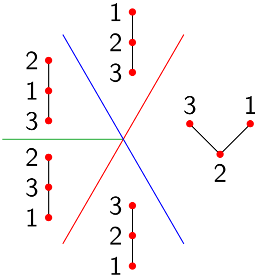
Generalized permutahedron: deformation of Π_n
i.e. P generalized permutahedron iff $\mathcal{N}_{\mathrm{P}}$ coarsens braid fan





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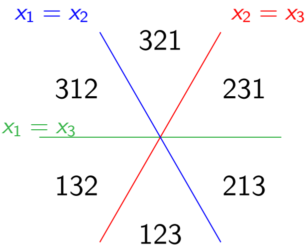
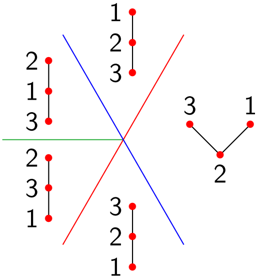
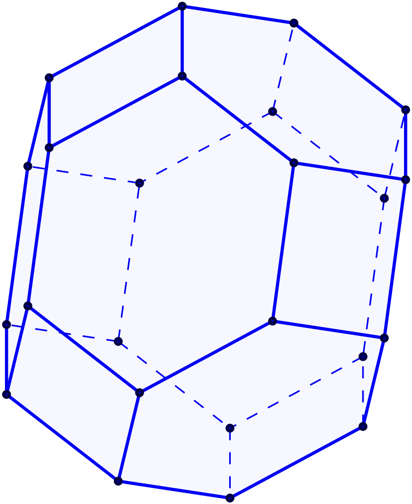
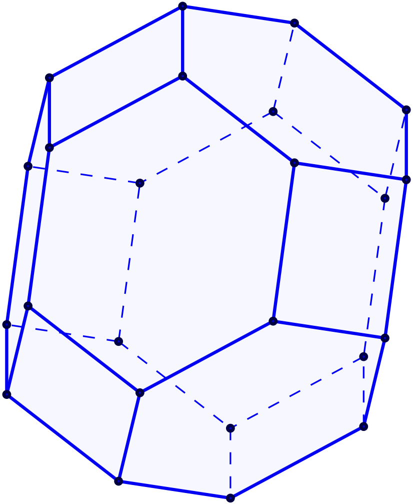




$\mathcal{P}(\mathrm{P})$: all the posets associated to faces of P



Permutahedron Π_4



Sequence of deformations of Π_4

Cone of deformations

Minkowski sum: $\mathsf{P} + \mathsf{Q} = \{\boldsymbol{p} + \boldsymbol{q} \ ; \ \boldsymbol{p} \in \mathsf{P}, \ \boldsymbol{q} \in \mathsf{Q}\}$

Theorem

If Q, R *deformations of* P, *then:*
- *for all* $\lambda > 0$, $\lambda\mathsf{Q}$ *deform. of* P
- $\mathsf{Q} + \mathsf{R}$ *deform. of* P

Definition

Deformation cone: $\mathbb{DC}(\mathsf{P}) := \{\mathsf{Q} \ ; \ \mathsf{Q} \text{ deformation of } \mathsf{P}\}$ is a cone.

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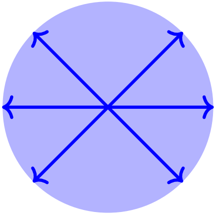

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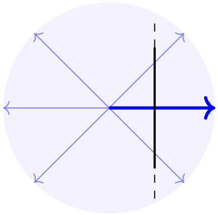

Cone of deformations

Minkowski sum: $\mathrm{P} + \mathrm{Q} = \{\boldsymbol{p} + \boldsymbol{q} \ ; \ \boldsymbol{p} \in \mathrm{P}, \ \boldsymbol{q} \in \mathrm{Q}\}$

Theorem

If Q, R *deformations of* P, *then:* *for all* $\lambda > 0$, λQ *deform. of* P
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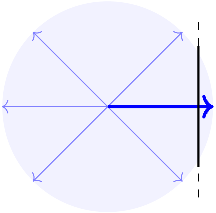

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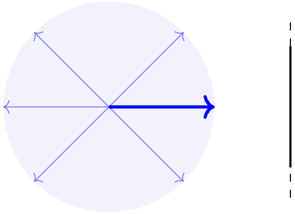

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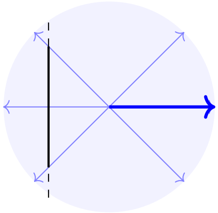

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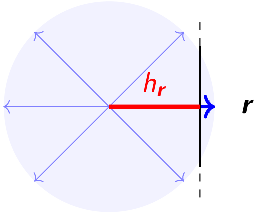

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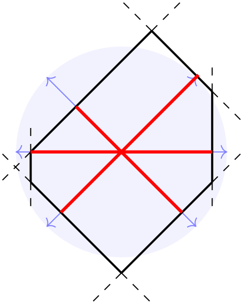

Parametrization:

height vector:
$\boldsymbol{h} = (h_{\boldsymbol{r}})_{\boldsymbol{r} \text{ rays}}$

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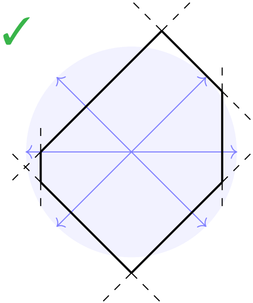

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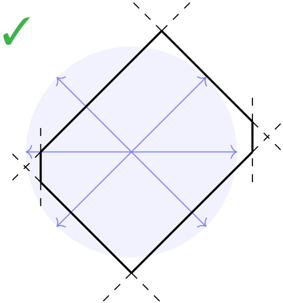

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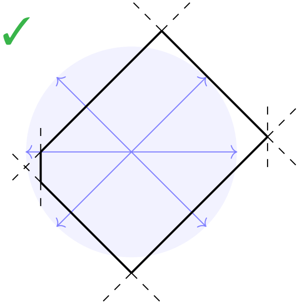

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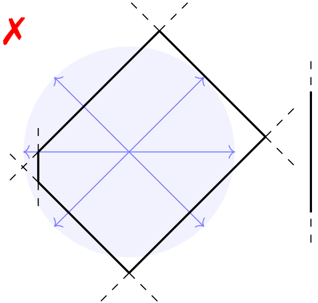

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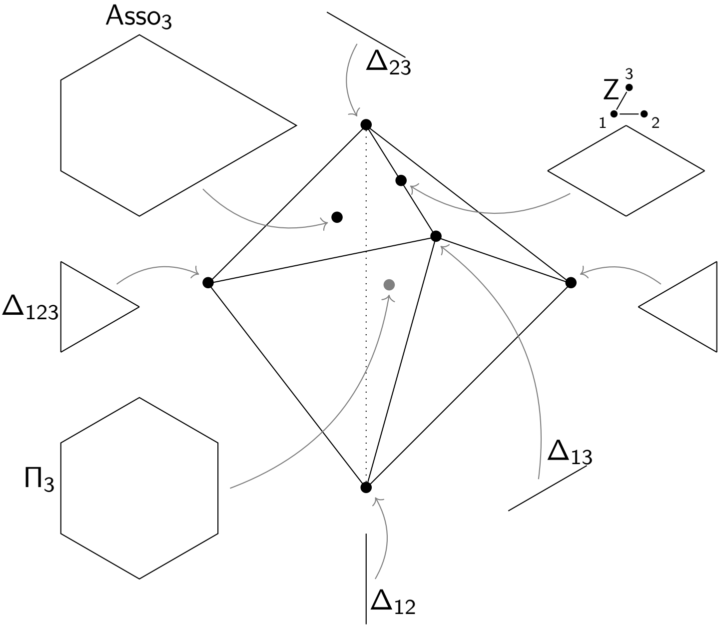
Definition

Submodular cone: deformation cone of the permutahedron Π_n

	$\mathbb{DC}(\Pi_n)$
Dim (no lineal)	$2^n - n - 1$
# facets	$\binom{n}{2}2^{n-2}$
# rays	unknown!

Submodular Cone for Π_3



Submodular Cone's faces

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		is simplicial!

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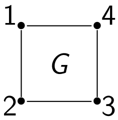
	$\mathbb{DC}(\Pi_n)$	$\mathbb{DC}(\mathrm{Asso}_n)$	$\mathbb{DC}(\mathrm{Z}_G)$	$\mathbb{DC}(\mathrm{N}_B)$
Dim (no lineal)	$2^n - n - 1$	$\binom{n}{2}$	N	N
# facets	$\binom{n}{2}2^{n-2}$	$\binom{n}{2}$	E	E
# rays	unknown!	$\binom{n}{2}$	X	X
		is simplicial!	T	T

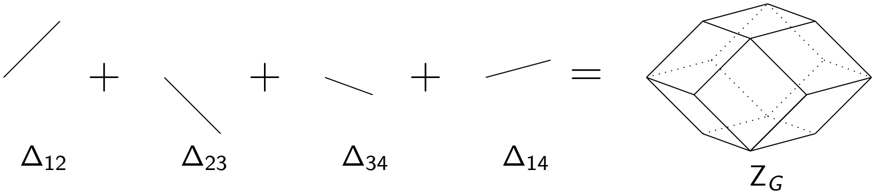
$G = (V, E)$ a graph, $n = |V|$

Definition

Graphical zonotope $\mathrm{Z}_G := \sum_{(i,j)\in E} [\boldsymbol{e}_i, \boldsymbol{e}_j]$

Z_G deformation of $\Pi_n \implies \mathbb{DC}(\mathrm{Z}_G)$ is a face of $\mathbb{DC}(\Pi_n)$





My contribution - Graphical Zonotopes

Theorem (Padrol, Pilaud, P., '23)

Explicit facet-description of $\mathbb{DC}(\mathsf{Z}_G)$

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Corollary

dim $\mathbb{DC}(\mathsf{Z}_G) = \#$ *cliques of* G

$\#$ *facets of* $\mathbb{DC}(\mathsf{Z}_G) = \sum_{(i,j)\in E} 2^{|\{k \ ; \ (i,k),(j,k)\in E\}|}$

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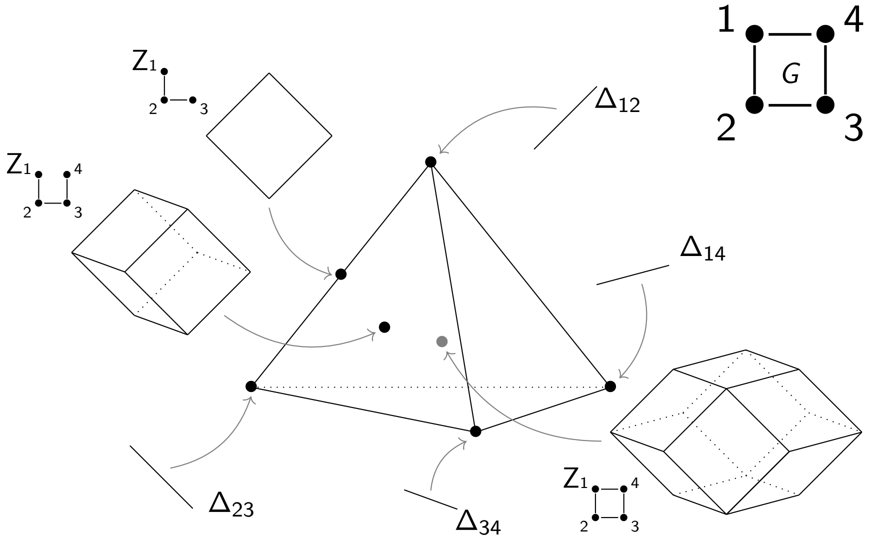
Corollary

$\mathbb{DC}(\mathsf{Z}_G)$ *simplicial iff* G *without triangle*

NB: Recover facet-description of $\mathbb{DC}(\Pi_n)$

My contribution - Graphical Zonotopes



My contribution - Nestohedra

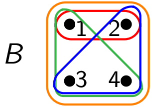
Definition

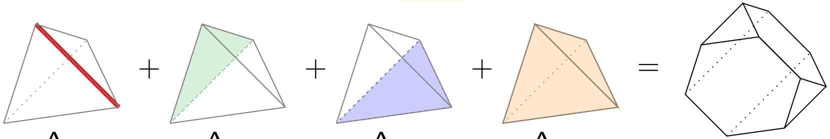
Building set $B \subseteq 2^{[n]}$ with: $X_{1,2} \in B, X_1 \cap X_2 \neq \emptyset \Rightarrow X_1 \cup X_2 \in B$

Definition

Nestohedron $\mathrm{N}_B := \sum_{X \in B} \Delta_X$ where $\Delta_X = \mathrm{conv}\{\boldsymbol{e}_i \ ; \ i \in X\}$

N_B deformation of $\Pi_n \implies \mathbb{DC}(\mathrm{N}_B)$ is a face of $\mathbb{DC}(\Pi_n)$





Δ_{12} Δ_{134} Δ_{234} Δ_{1234} N_B

My contribution - Nestohedra

Elementary blocks $X \in \varepsilon(B)$ iff X is not a union
Maximal block $\mu(X) := \max\{Y \in B\ ;\ Y \subsetneq X\}$

Theorem (Padrol, Pilaud, P., '23)

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Corollary

$\dim \mathbb{DC}(\mathsf{N}_B) = |B| - \#$ *singletons*

$\#$ *facets of* $\mathbb{DC}(\mathsf{N}_B) = |\varepsilon(B)| + \sum_{X \in B \smallsetminus \varepsilon(B)} \binom{|\mu(X)|}{2}$

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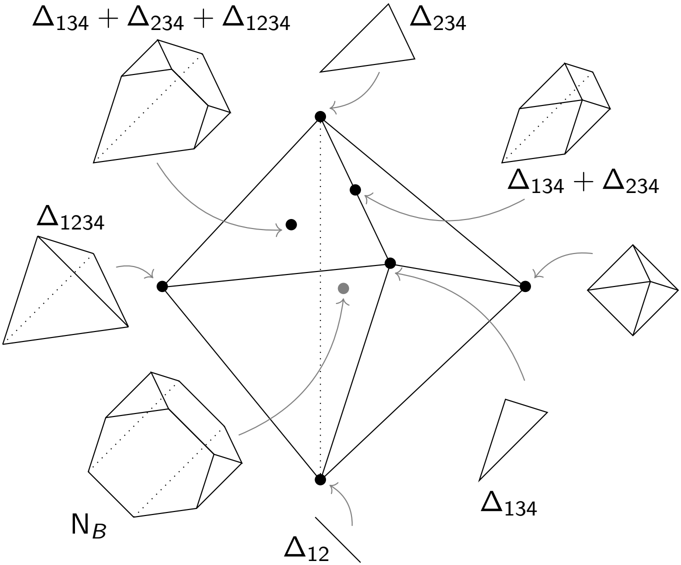
Corollary

$\mathbb{DC}(\mathsf{N}_B)$ *simplicial iff* B *has no non-elementary block with 3 maximal subblocks*

NB: Recover facet-description of $\mathbb{DC}(\Pi_n)$

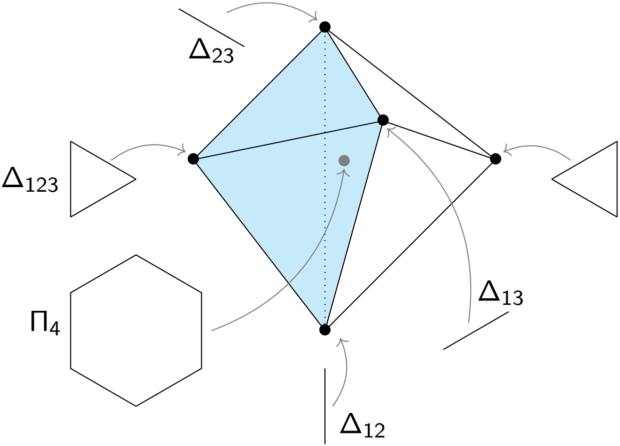
My contribution - Nestohedra



Ongoing work - Hypergraphic polytopes

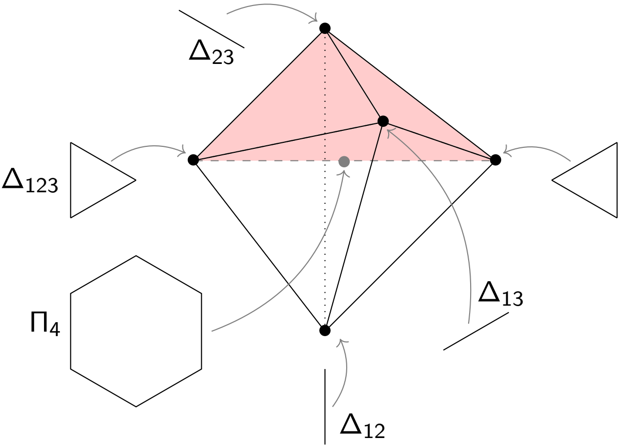
Definition

Hypergraphic pol $\mathsf{P}_H := \sum_{X \in H} \Delta_X \quad$ with $H \subseteq 2^{[n]}$

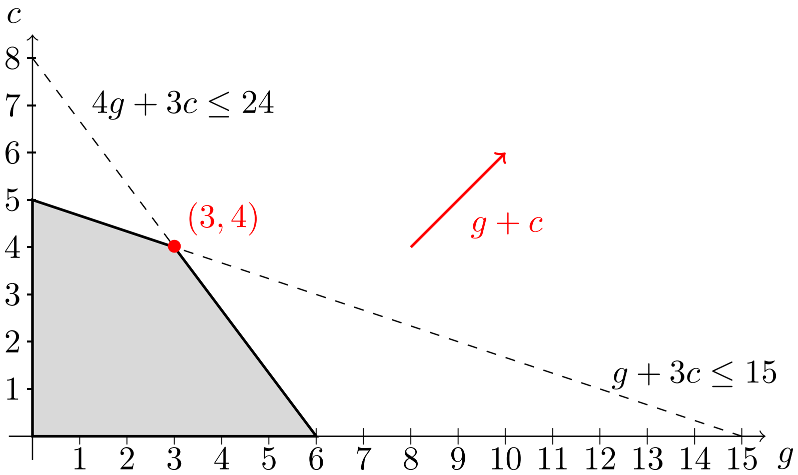


Definition

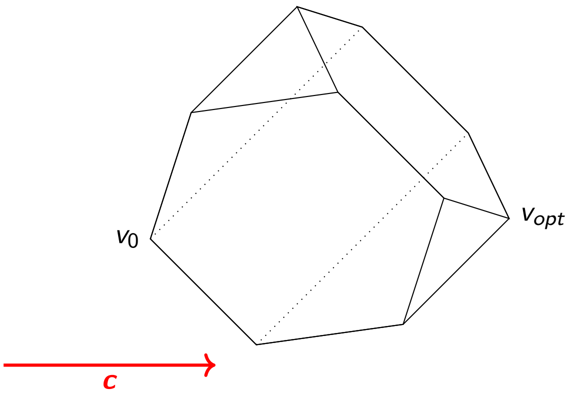
Quotientopes: Minkowski sum of shard polytopes



Max-slope Pivot Polytopes

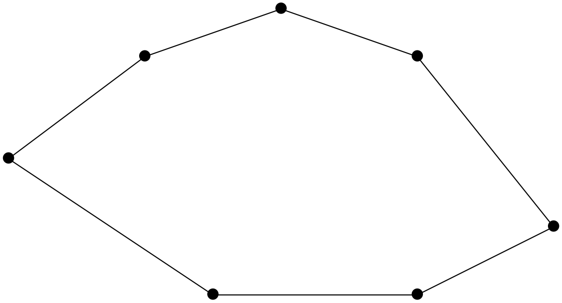

c
8
7
6
5
4
3
2
1
$4g + 3c \leq 24$
$(3, 4)$
$g + c$
$g + 3c \leq 15$
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
g

Simplex method



Max-slope pivot rule

Linear optimization in dimension 2 (simplex method):



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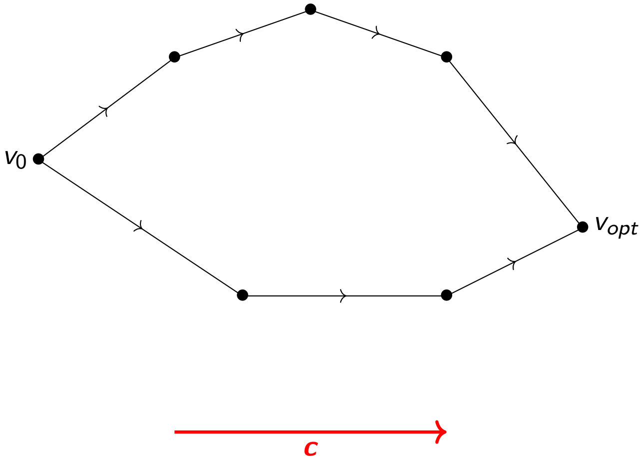

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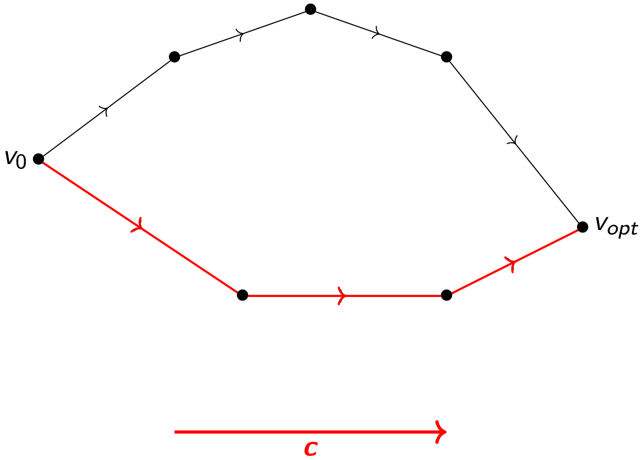

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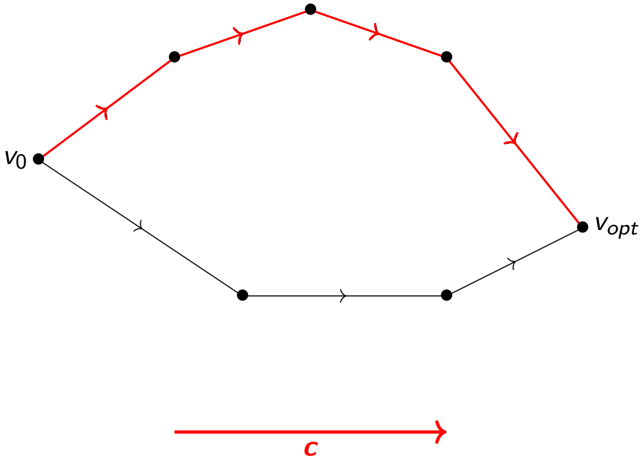
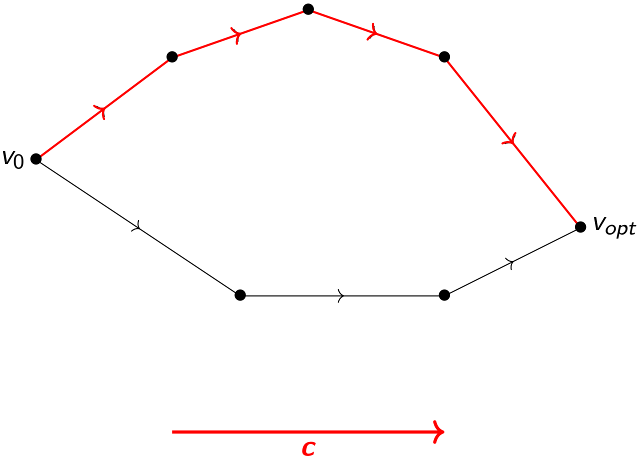


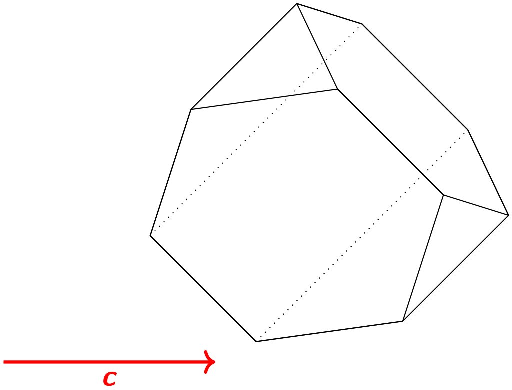
Max-slope pivot rule

Linear optimization in dimension 2 (simplex method): **EASY** !



Convention: choose upper

Optimization in higher dimension: make it 2-dimensional !



Max-slope pivot rule

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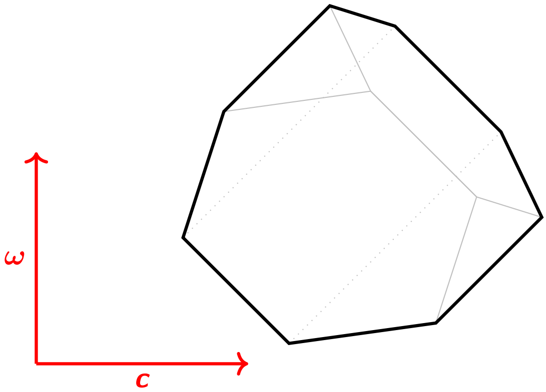

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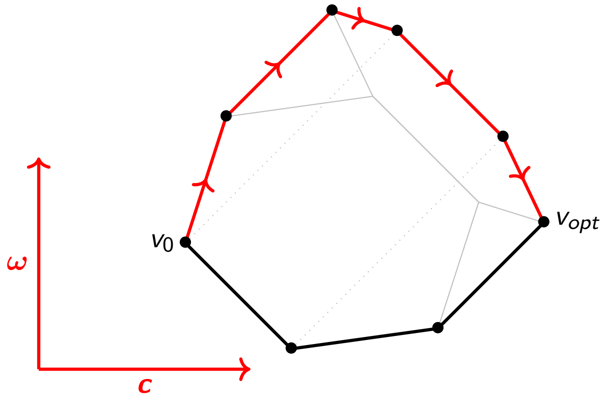

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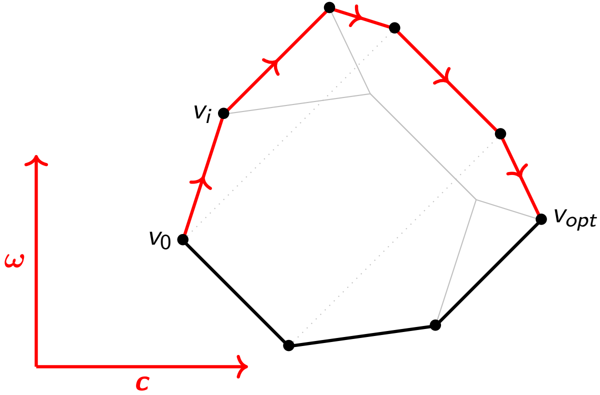

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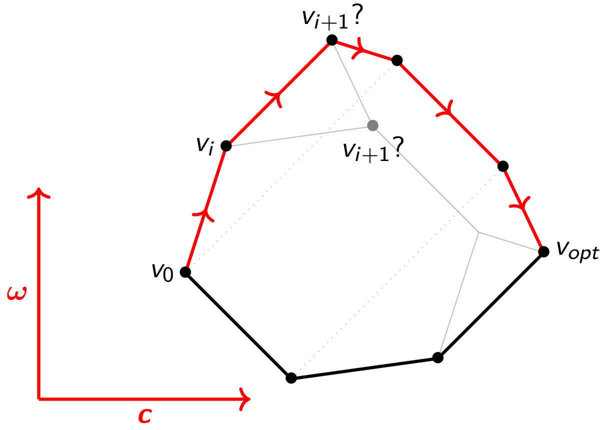

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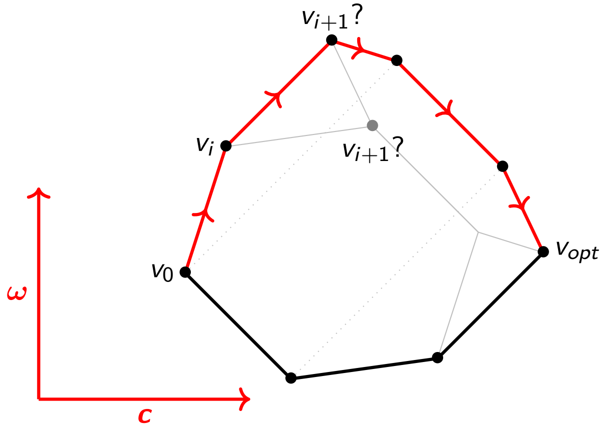
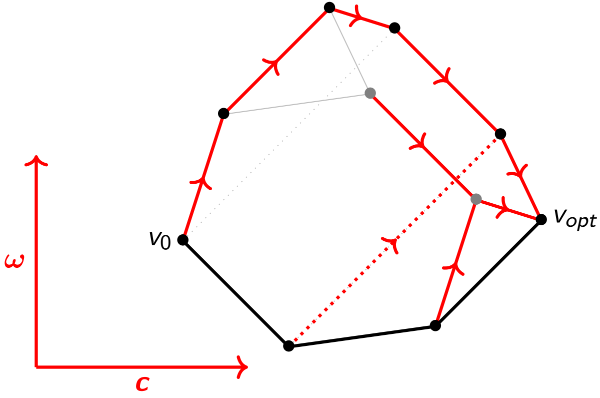


Max-slope pivot rule: take (improving) neighbor with best slope

Max-slope pivot rule

Optimization in higher dimension: make it 2-dimensional !



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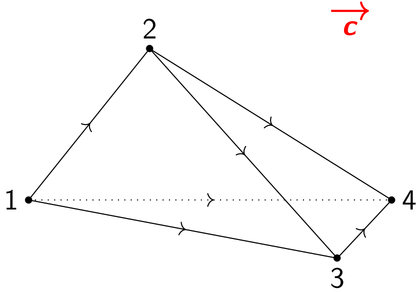
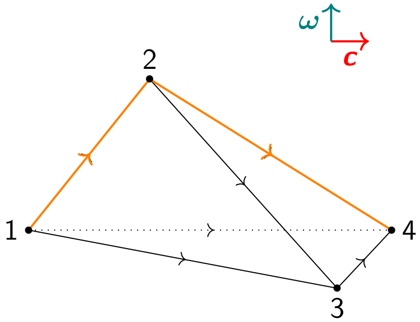


2
$\overrightarrow{c}$
1
4
3


ω


ω
c
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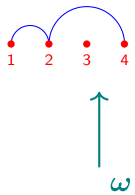
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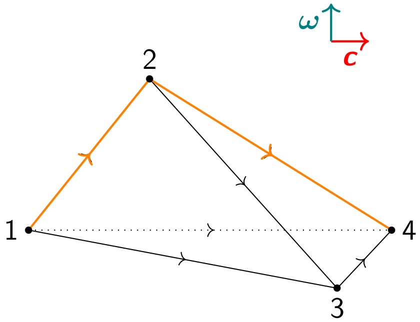
ω
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1
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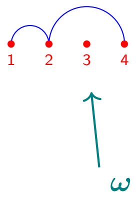
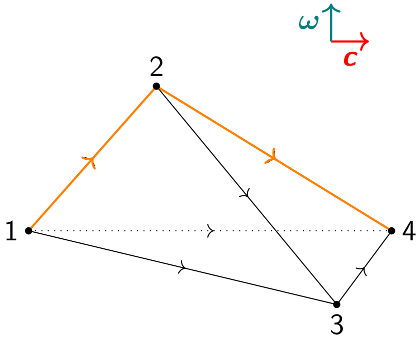
4

Coherent paths of the d-simplex





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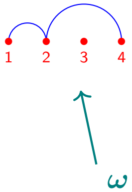

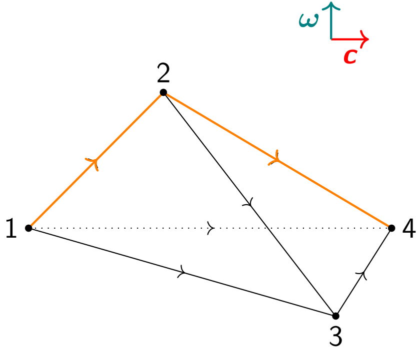

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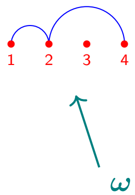

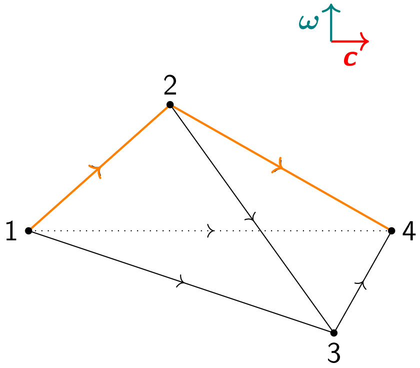

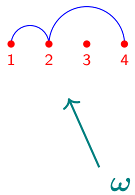
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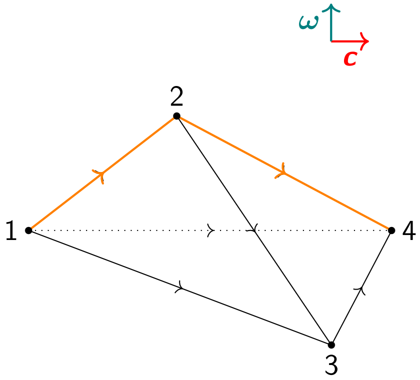
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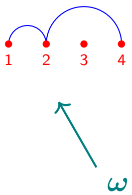
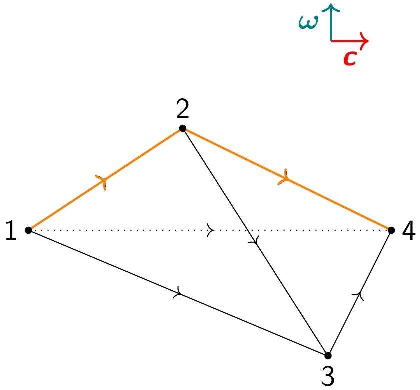
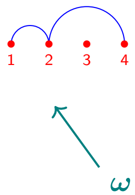
Coherent paths of the d-simplex



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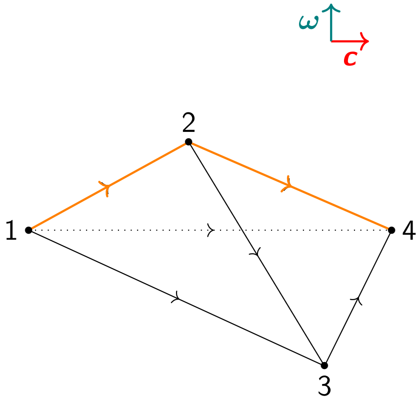
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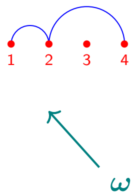
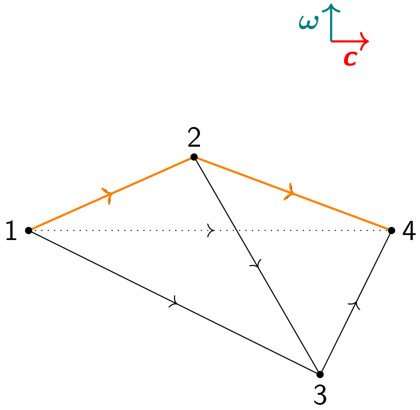
Coherent paths of the d-simplex





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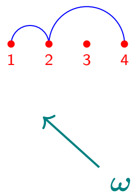

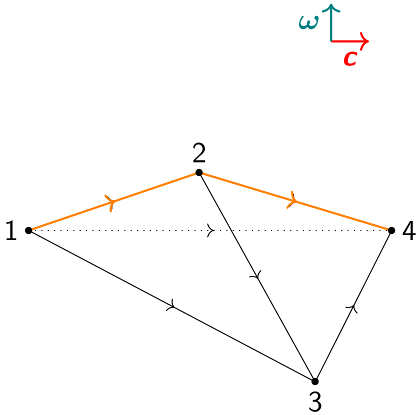

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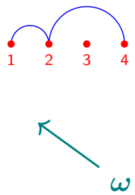

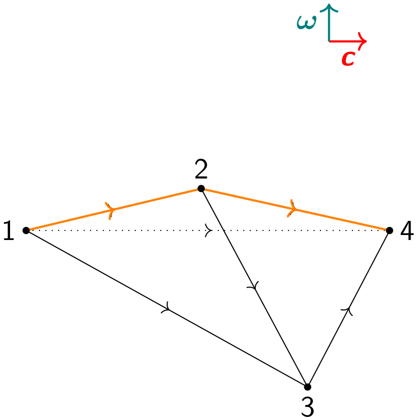

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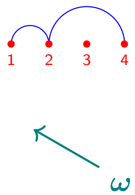

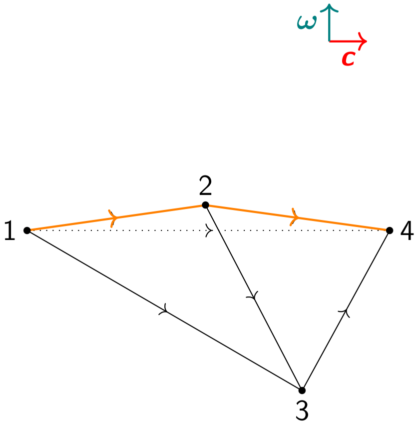

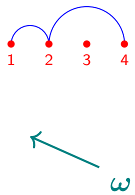
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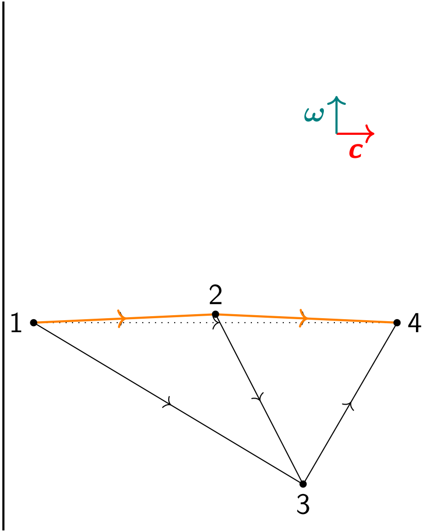
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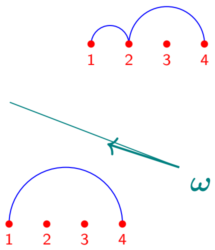
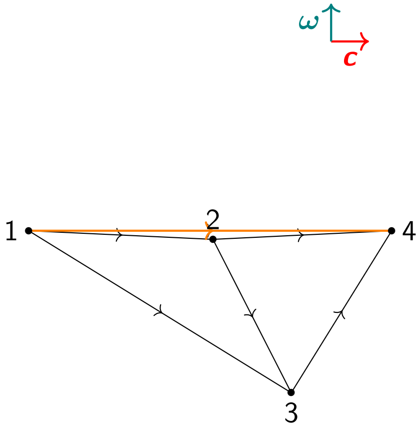
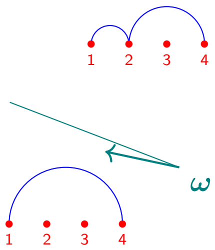
Coherent paths of the d-simplex



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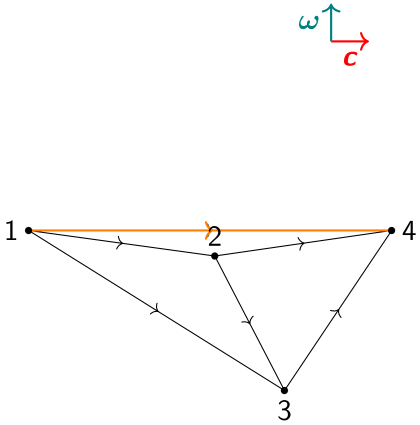
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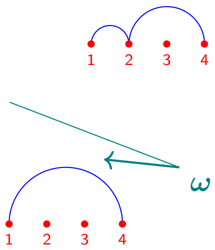
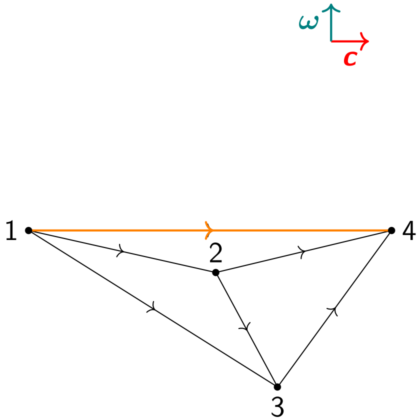
Coherent paths of the d-simplex





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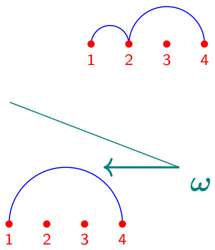

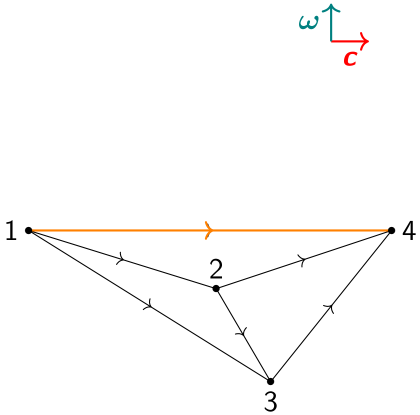

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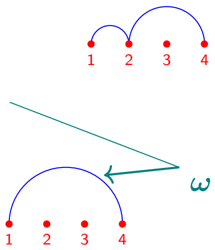

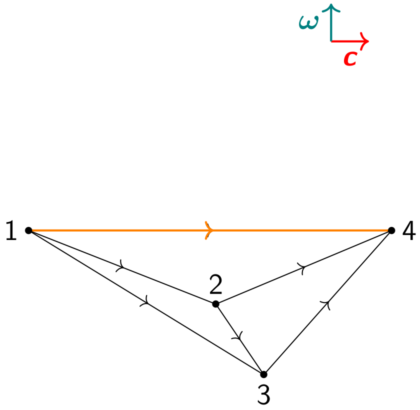

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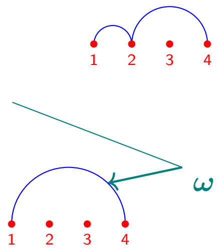

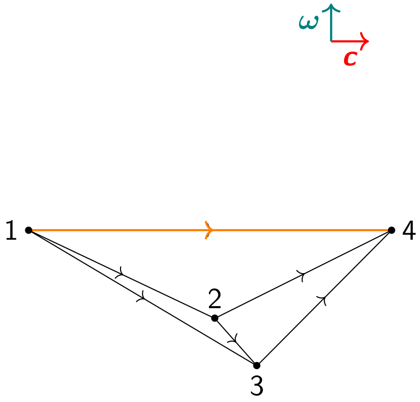

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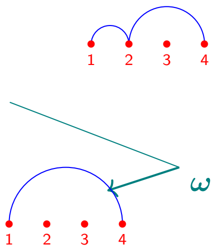

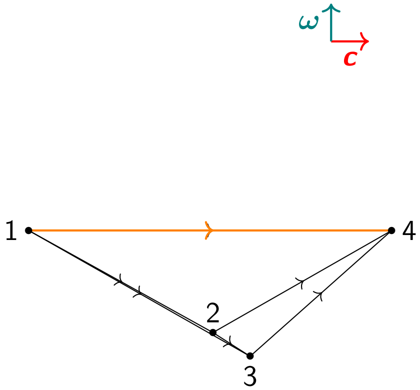

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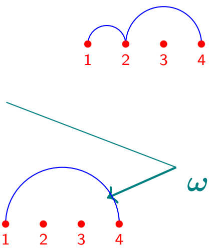

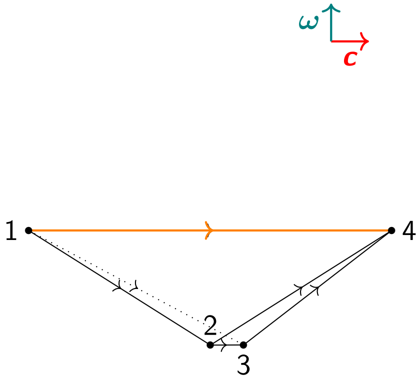

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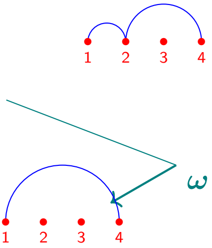

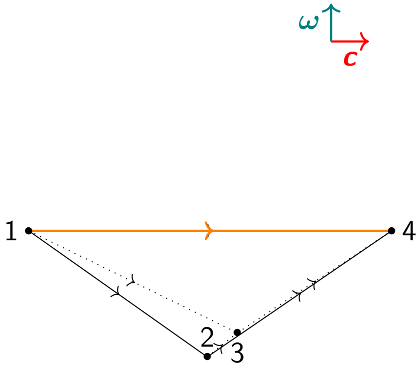

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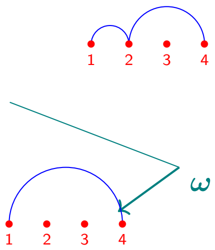

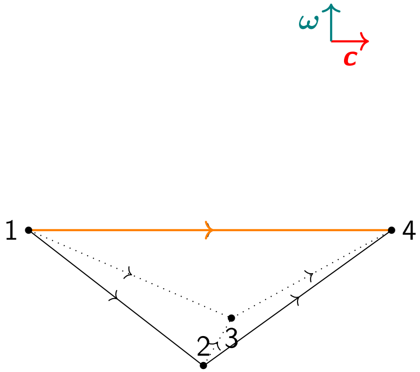

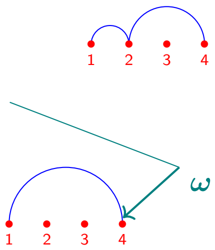
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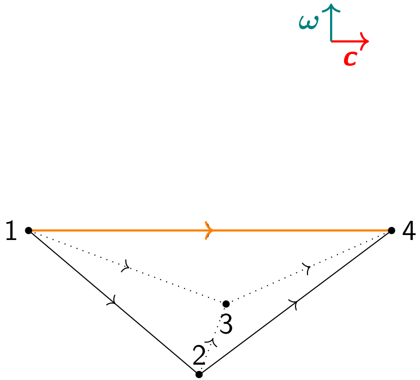
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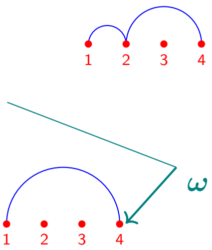
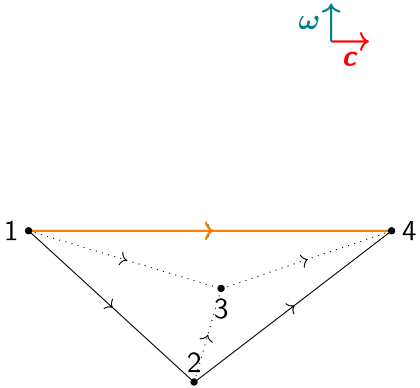
Coherent paths of the d-simplex





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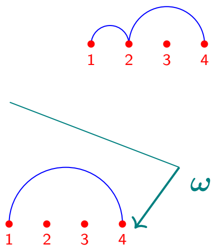

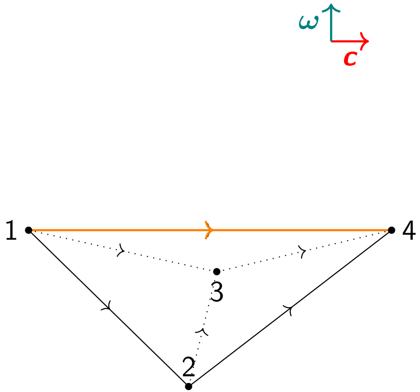

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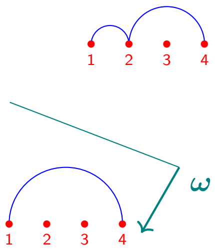

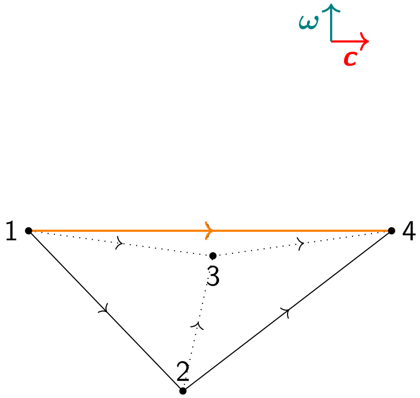

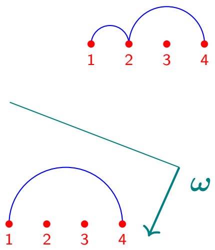

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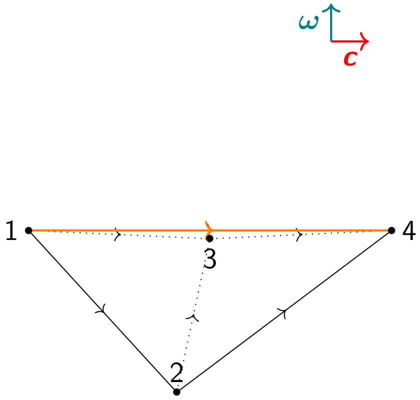

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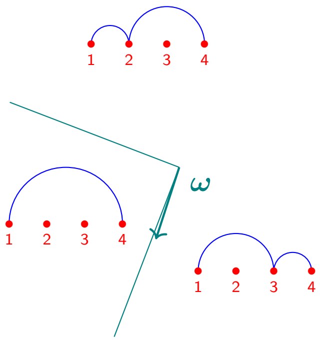
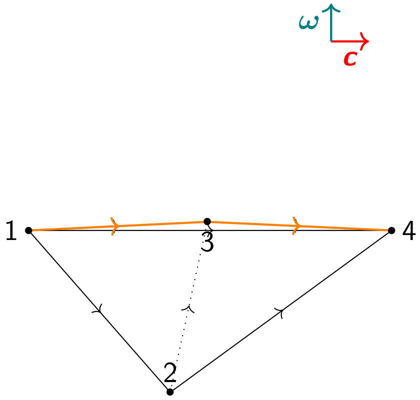
Coherent paths of the d-simplex





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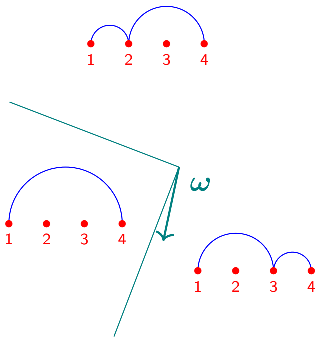

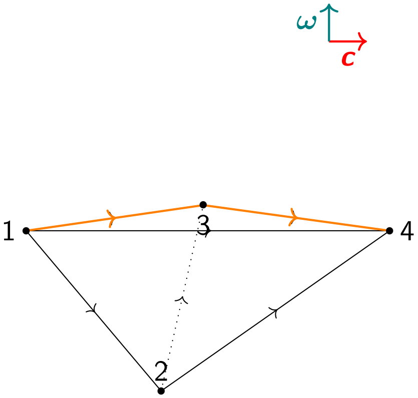

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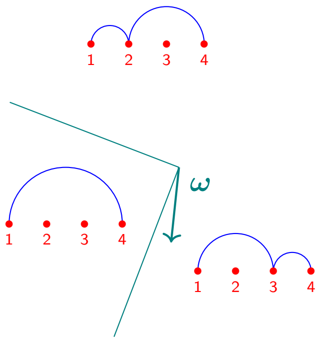

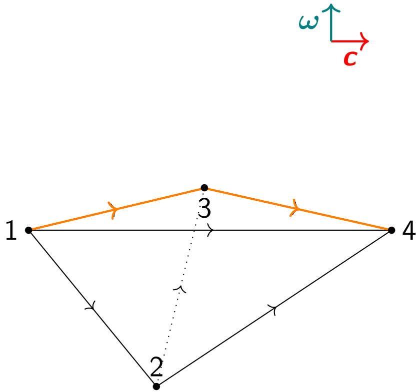

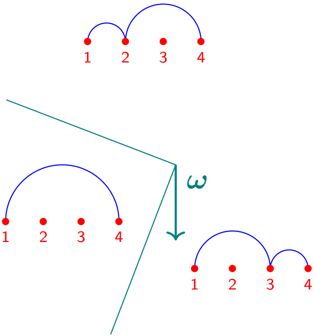
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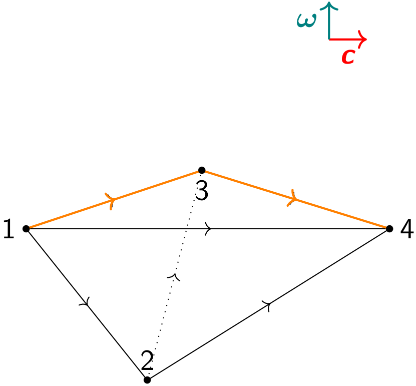
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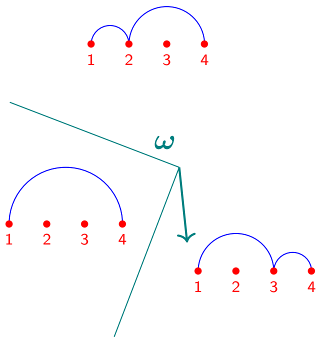
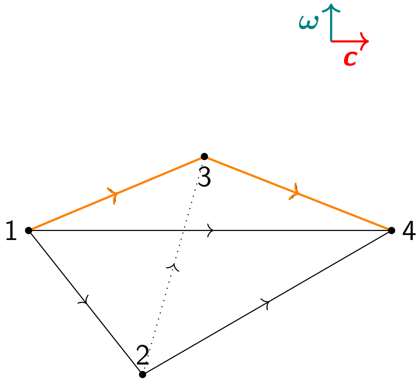
Coherent paths of the d-simplex





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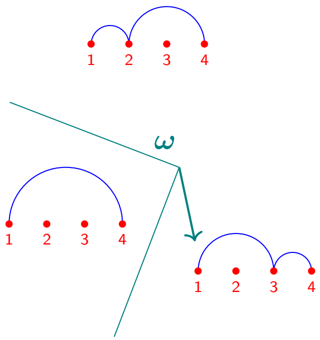

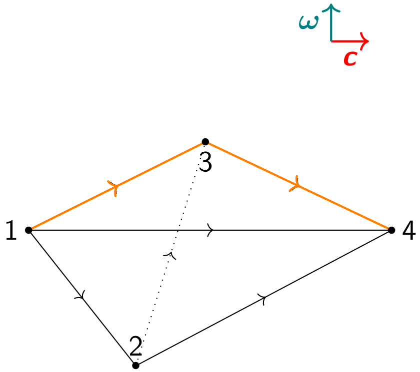

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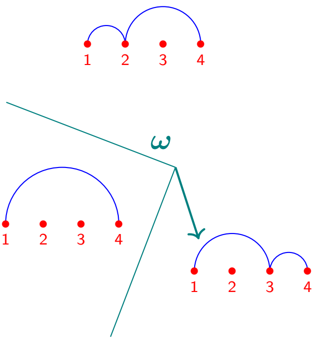

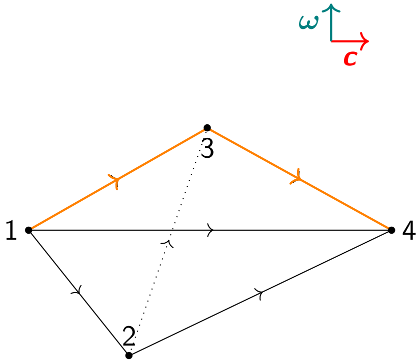

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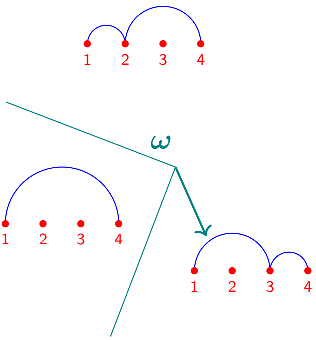

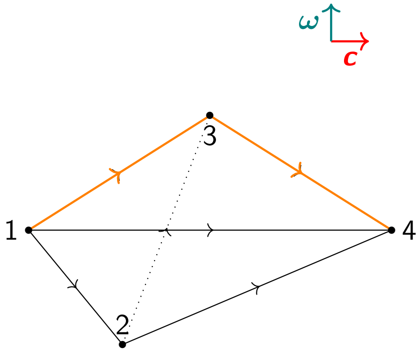

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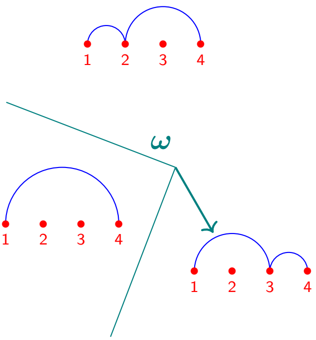

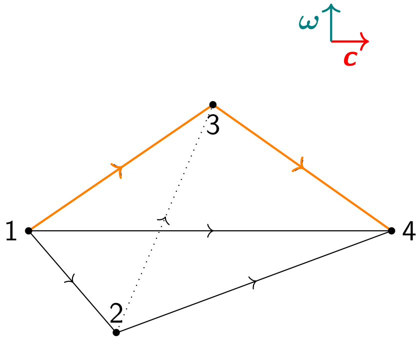

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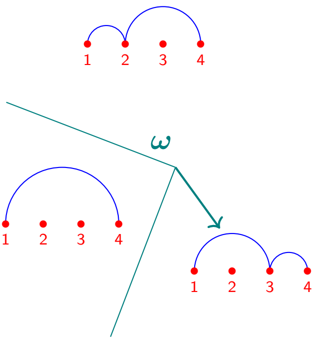

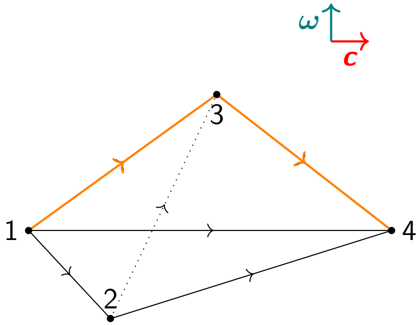

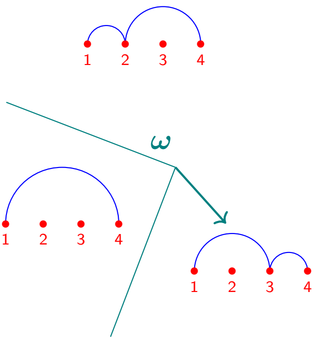
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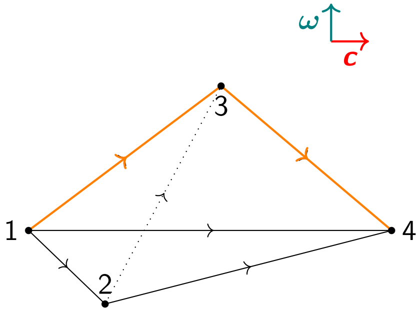
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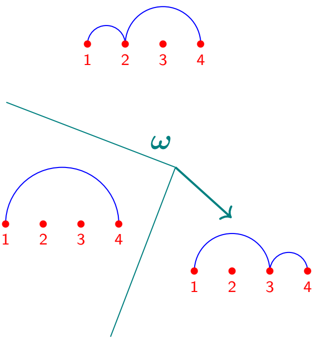
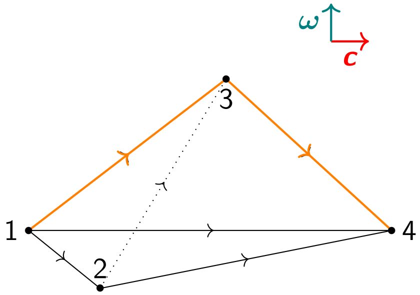
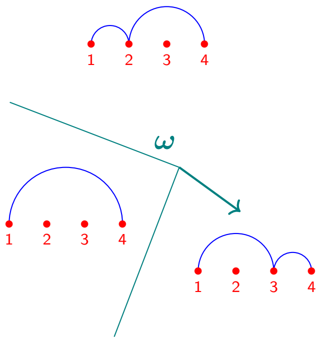
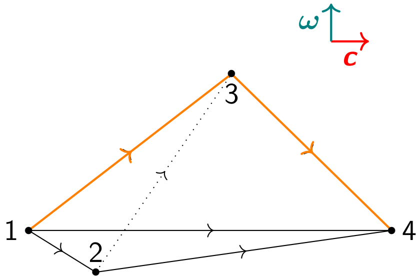
Coherent paths of the d-simplex



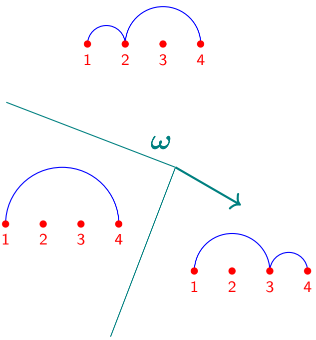


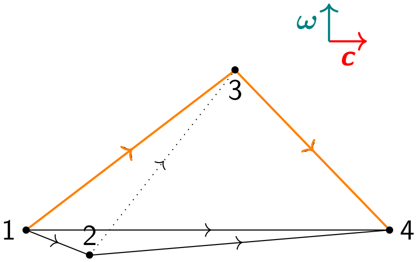
Coherent paths of the d-simplex



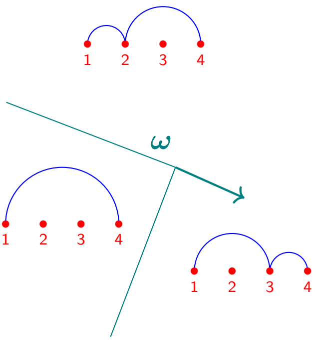


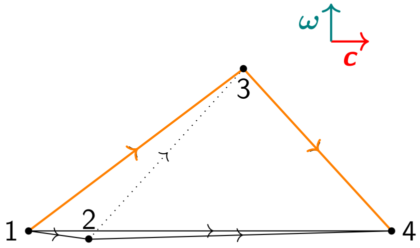
Coherent paths of the d-simplex



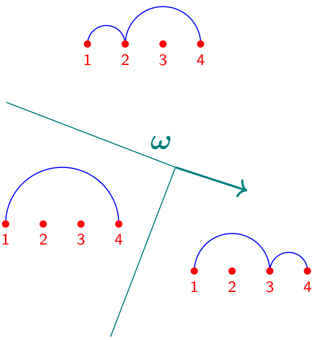


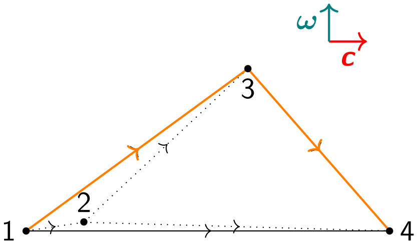
Coherent paths of the d-simplex

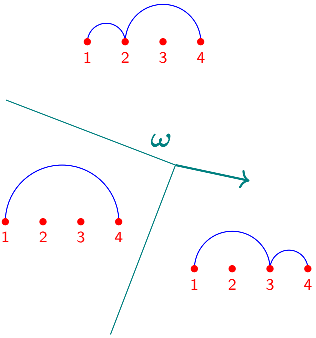




Coherent paths of the d-simplex



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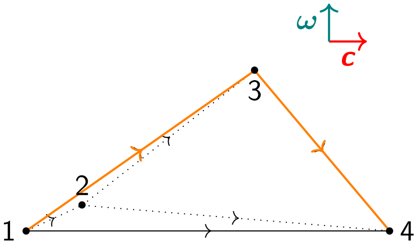
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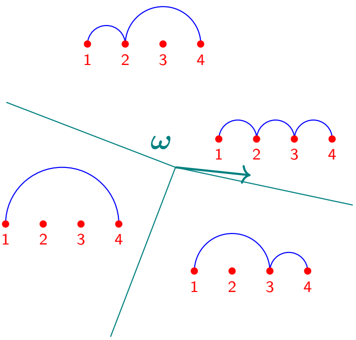
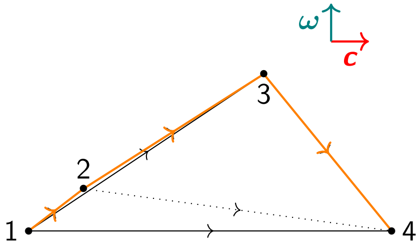
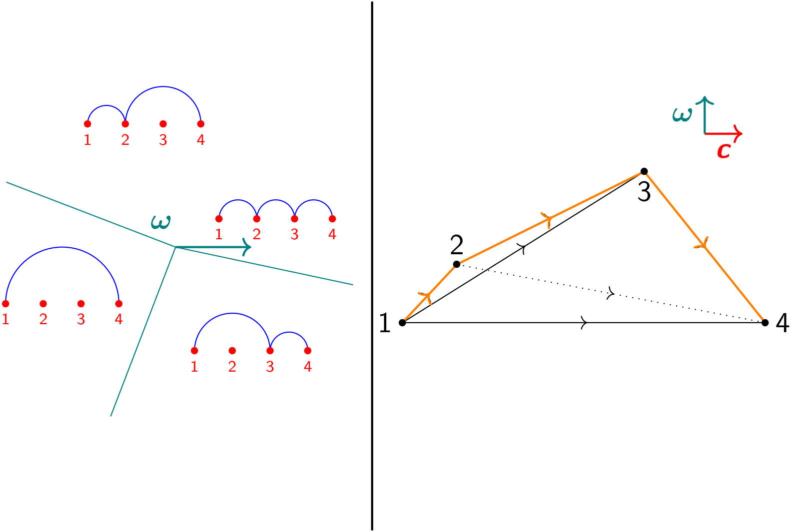
Coherent paths of the d-simplex

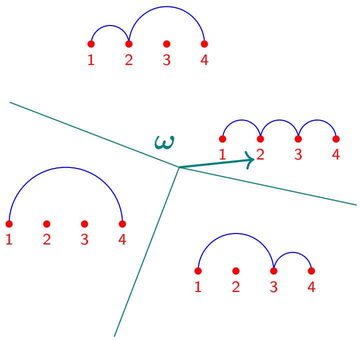


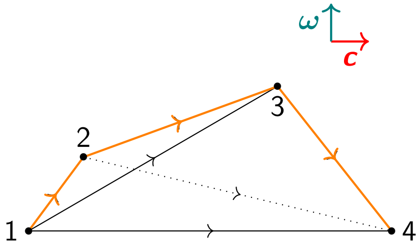


Coherent paths of the d-simplex

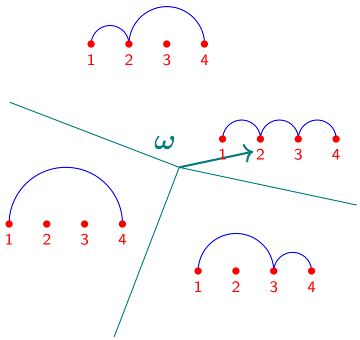


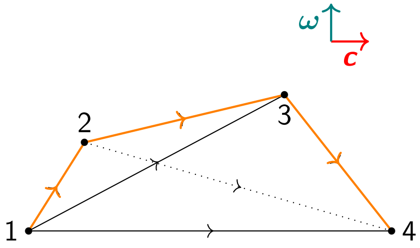
Coherent paths of the d-simplex



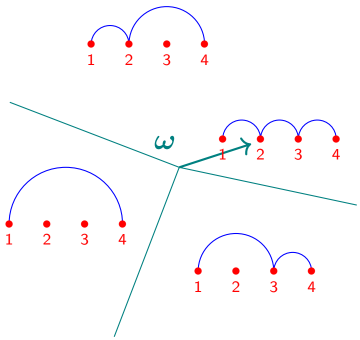


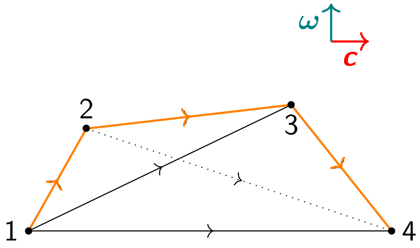
Coherent paths of the d-simplex



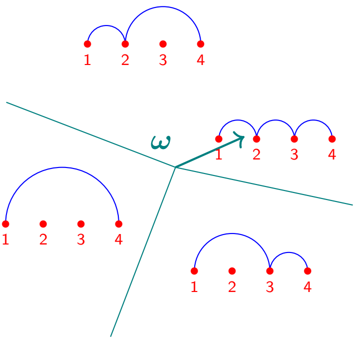


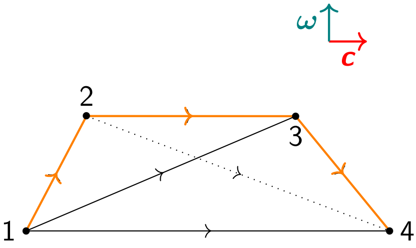
Coherent paths of the d-simplex



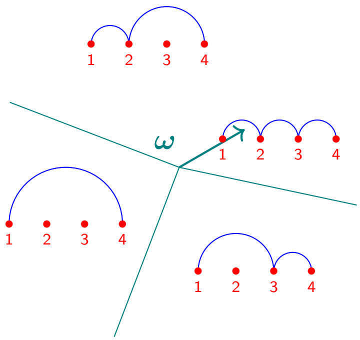


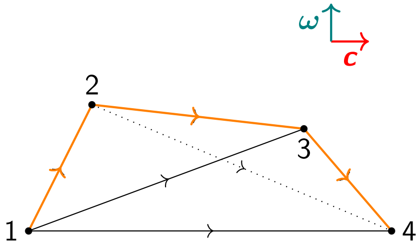
Coherent paths of the d-simplex



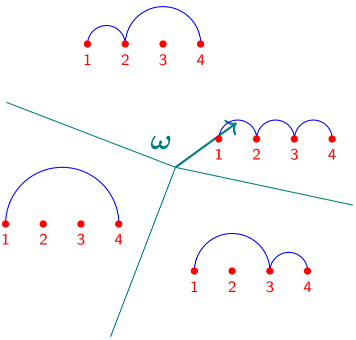


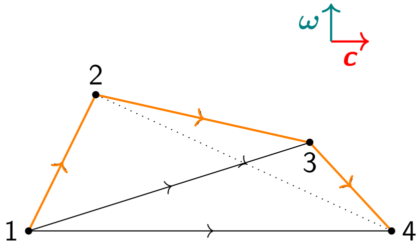
Coherent paths of the d-simplex

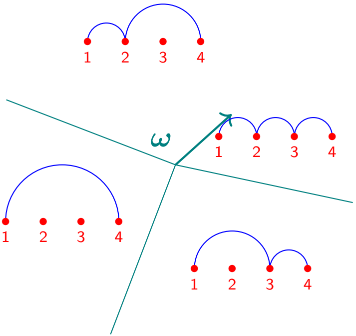




Coherent paths of the d-simplex







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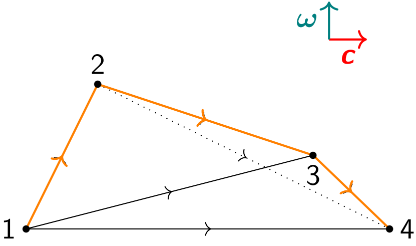

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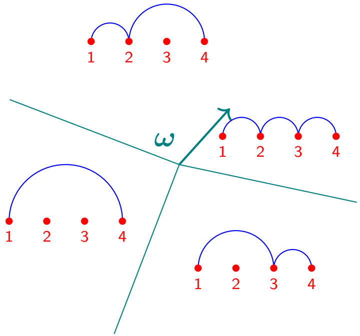
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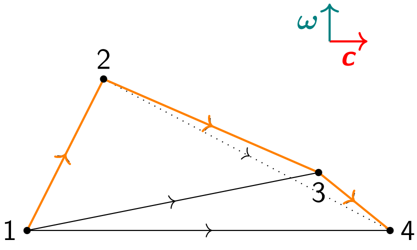
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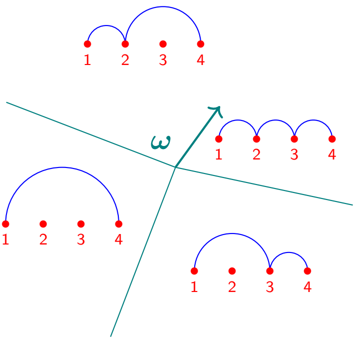
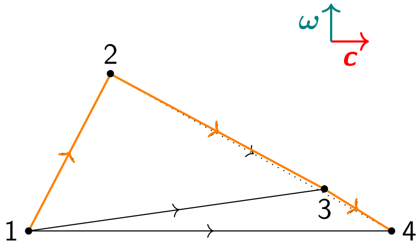
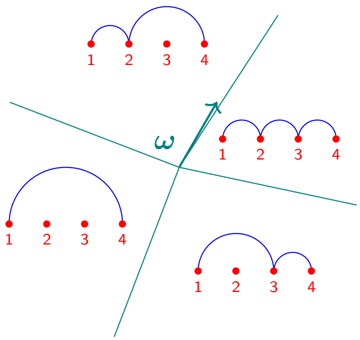
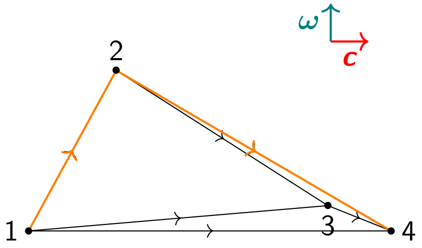
Coherent paths of the d-simplex



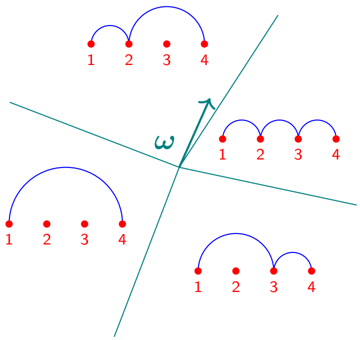


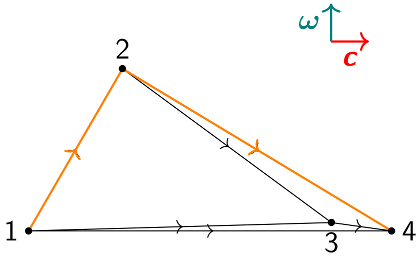
Coherent paths of the d-simplex

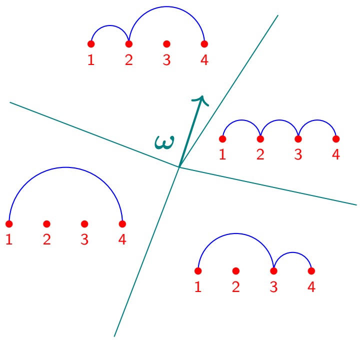




Coherent paths of the d-simplex



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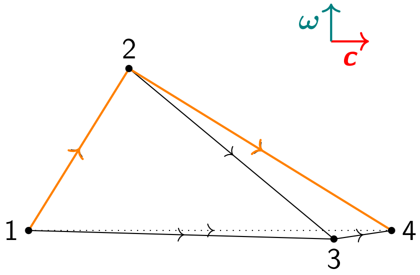
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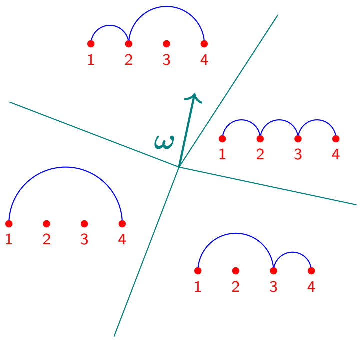
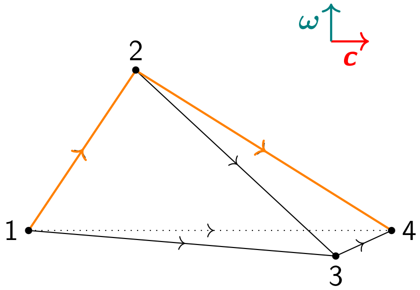
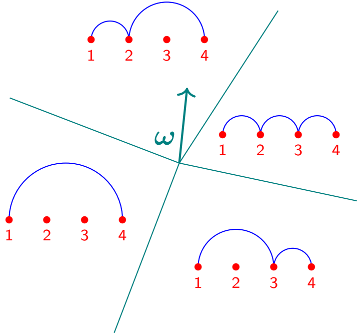
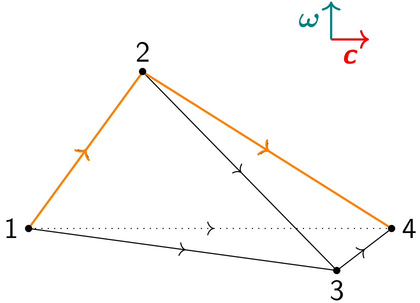
Coherent paths of the d-simplex



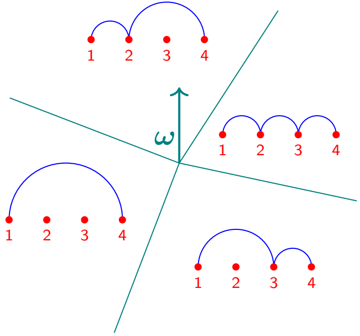


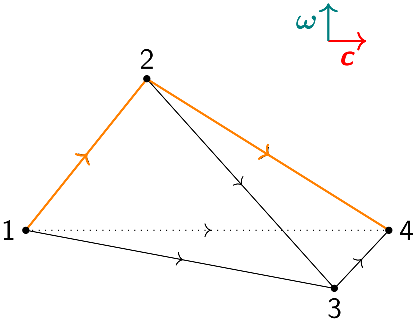
Coherent paths of the d-simplex



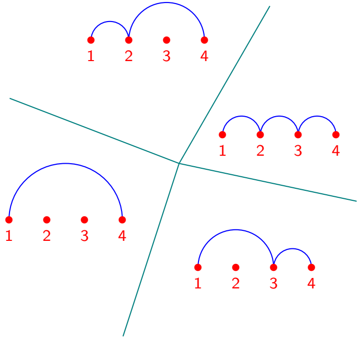


Coherent paths of the d-simplex





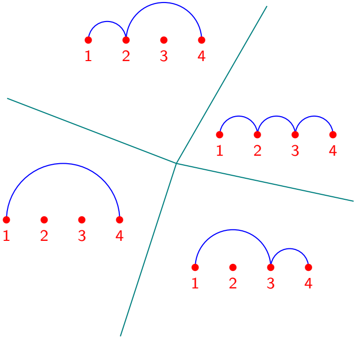
Monotone path polytope



Coherent monotone path: path obtained via max-slope pivot rule

Monotone path fan: Fan with $\omega \sim \omega'$ iff same path

Monotone path polytope



Coherent monotone path: path obtained via max-slope pivot rule

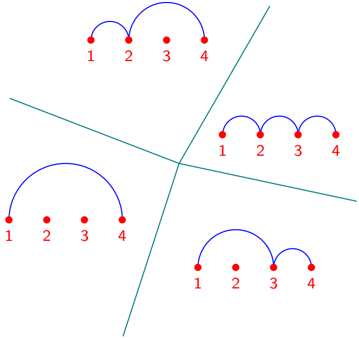
Monotone path fan: Fan with $\omega \sim \omega'$ iff same path

Theorem (Billera, Sturmfels, '92)

The monotone path fan is polytopal.

Monotone path polytope $\Sigma_c(P)$: dual to monotone path fan

Monotone path polytope



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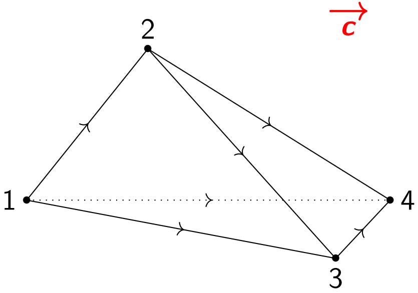
Monotone path polytope $\Sigma_{\boldsymbol{c}}(\mathrm{P})$: dual to monotone path fan

$$\Sigma_{\boldsymbol{c}}(\Delta_d) = \mathrm{Cube}_{d-1} \qquad \Sigma_{\boldsymbol{c}}(\mathrm{Cube}_d) = \Pi_d$$

Coherent arborescences of the d-simplex

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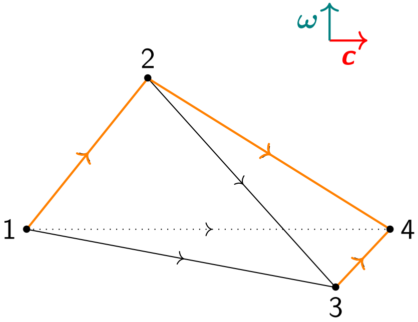
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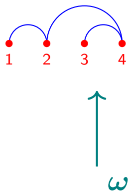
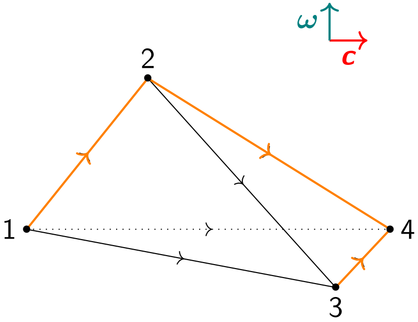
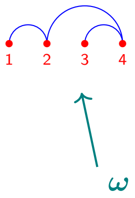
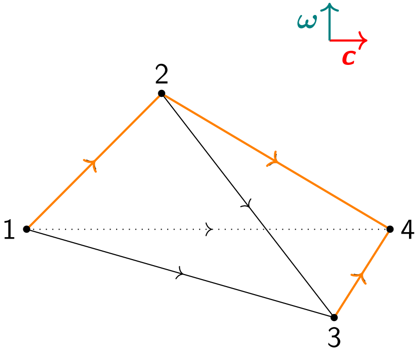
Coherent arborescences of the d-simplex

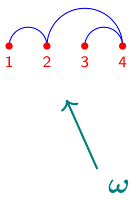




Coherent arborescences of the d-simplex



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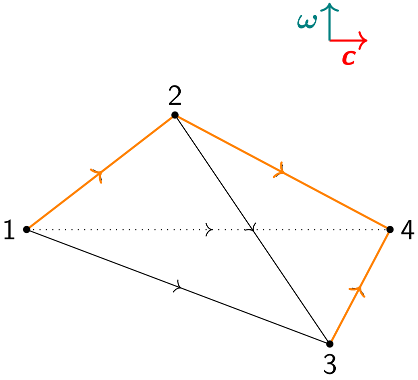
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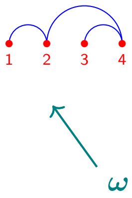
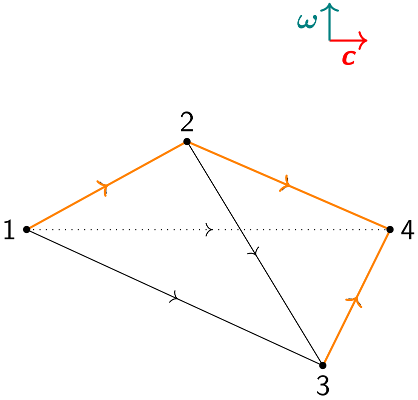
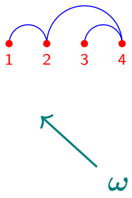
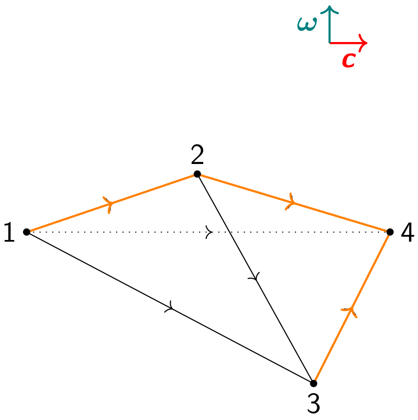
Coherent arborescences of the d-simplex

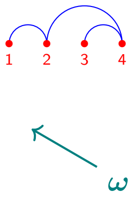




Coherent arborescences of the d-simplex



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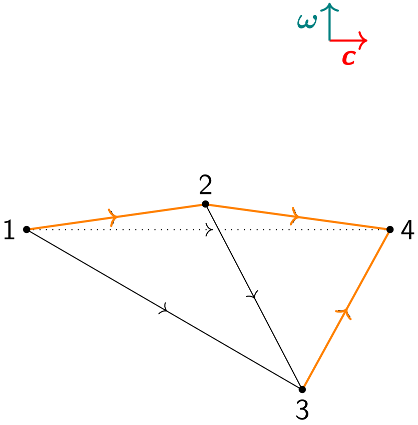
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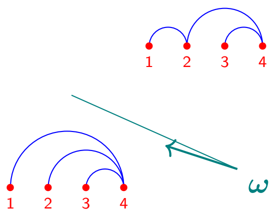
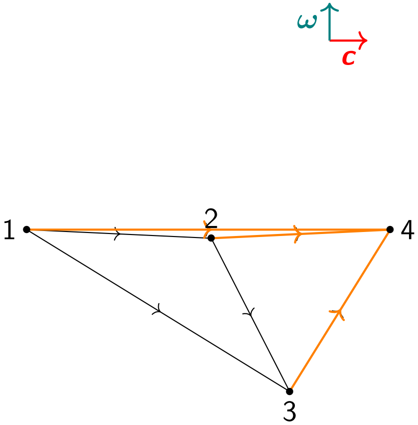
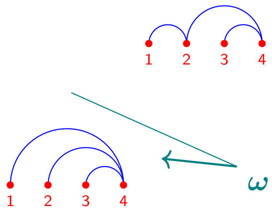
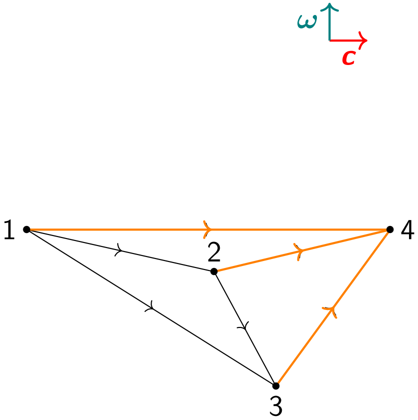
Coherent arborescences of the d-simplex



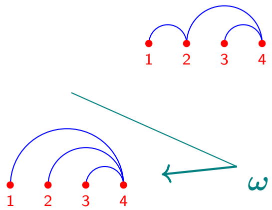


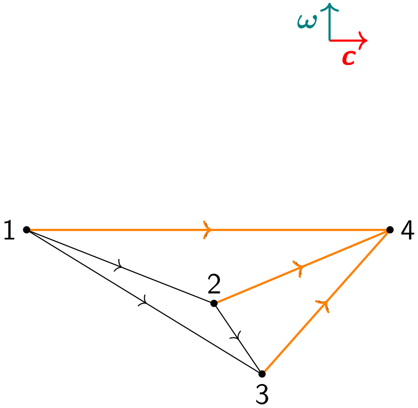
Coherent arborescences of the d-simplex

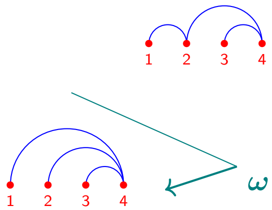




Coherent arborescences of the d-simplex



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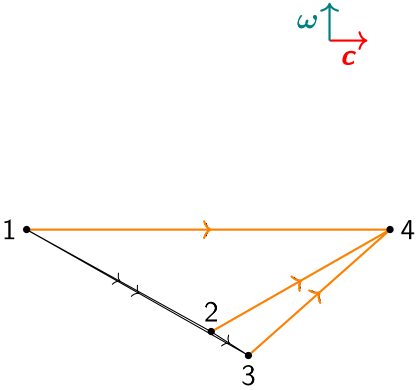
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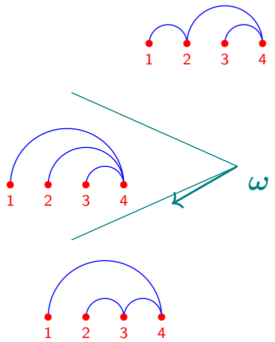
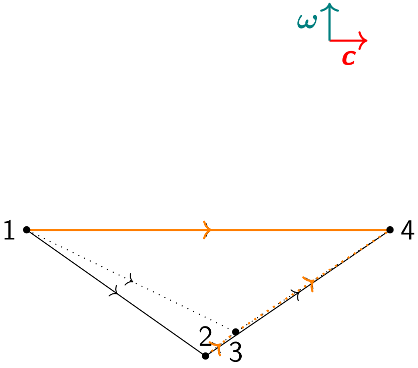
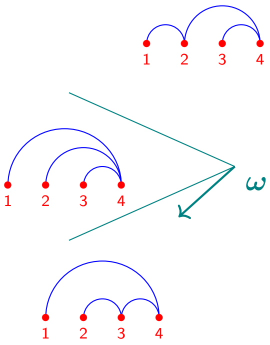
Coherent arborescences of the d-simplex



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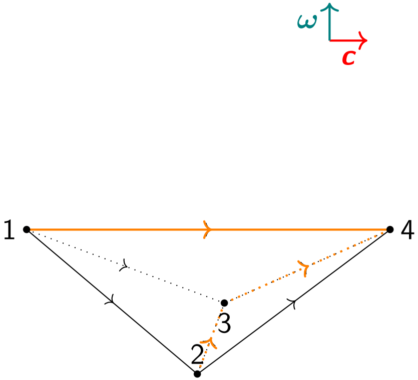
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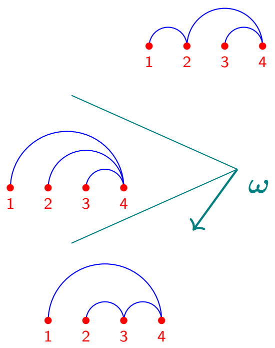
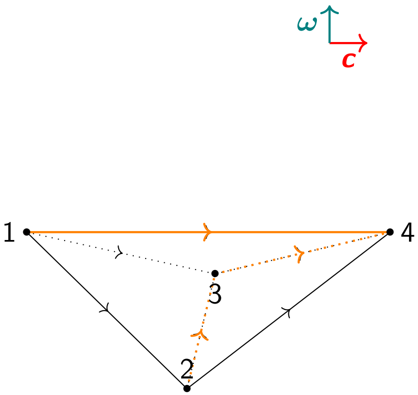
Coherent arborescences of the d-simplex





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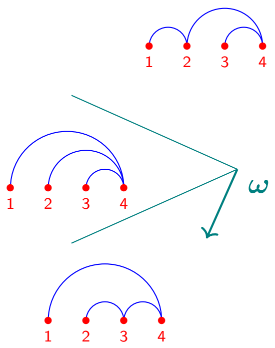

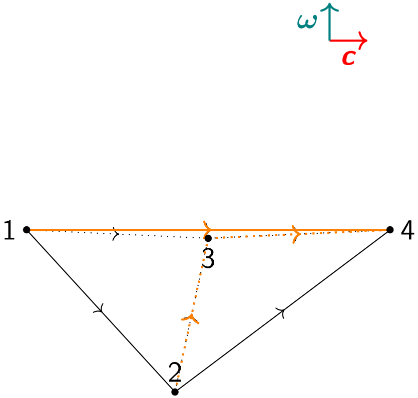

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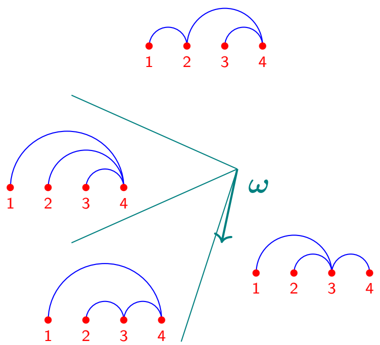

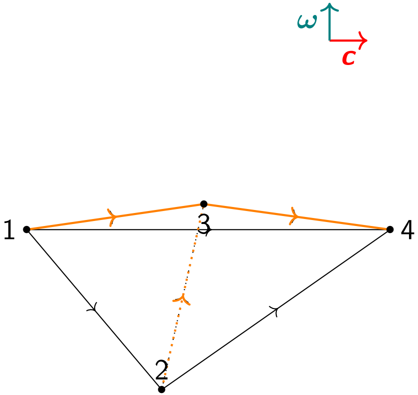

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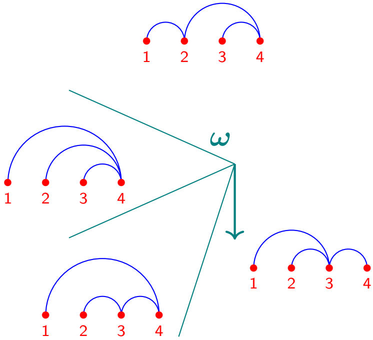

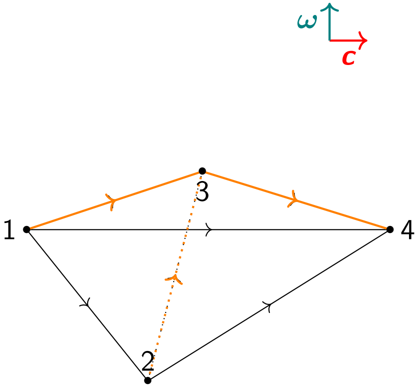

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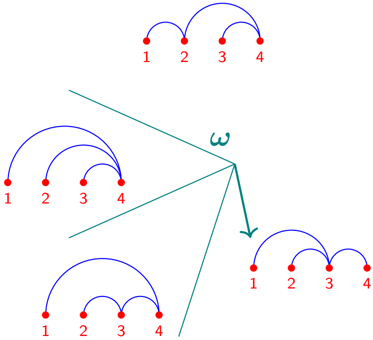

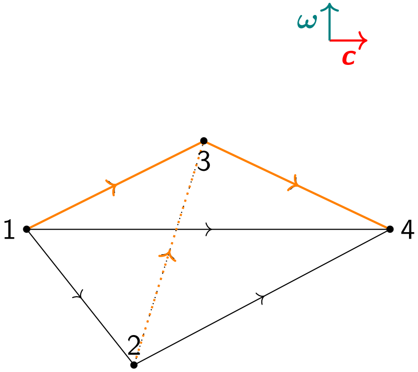

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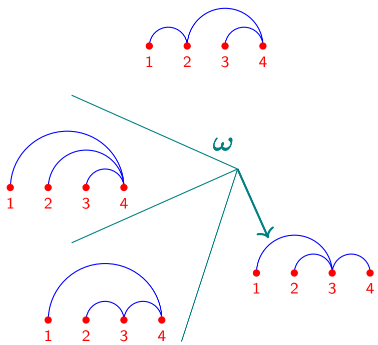

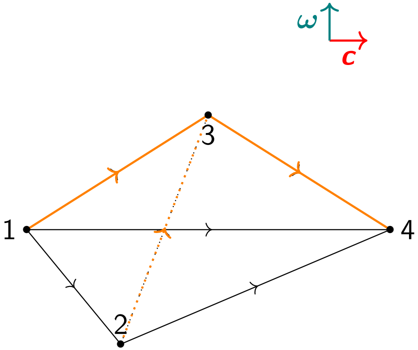

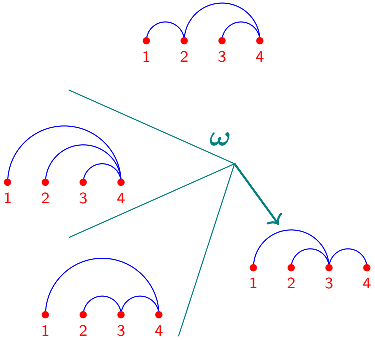
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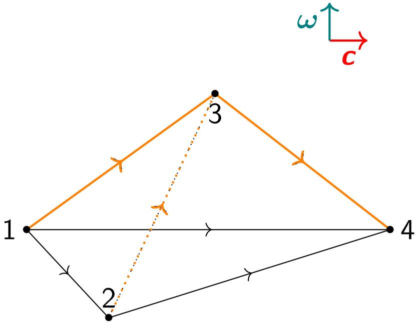
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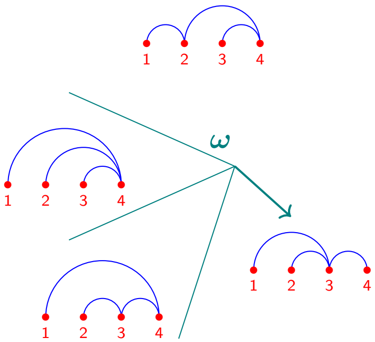
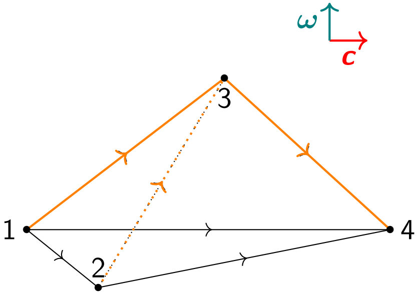
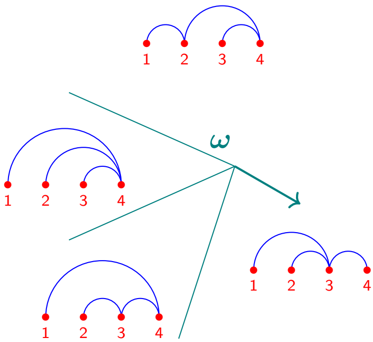
Coherent arborescences of the d-simplex



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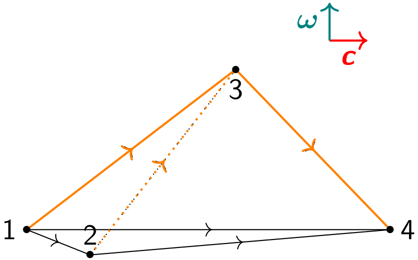
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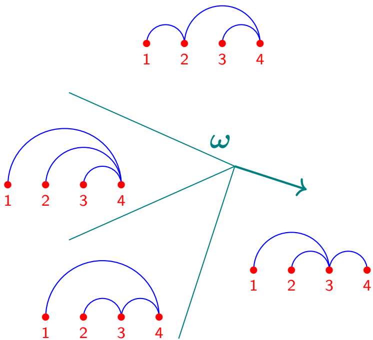
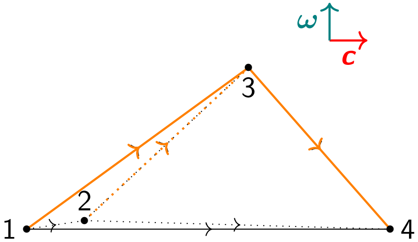
Coherent arborescences of the d-simplex





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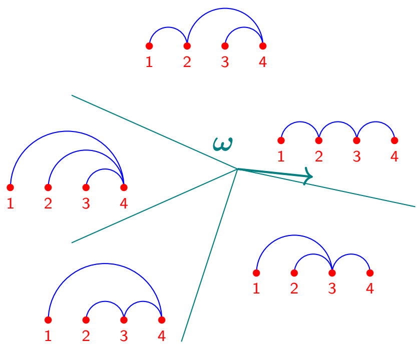

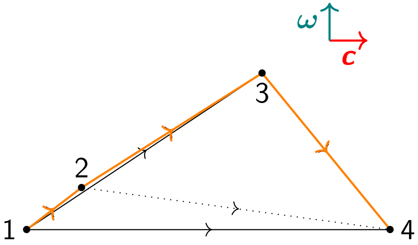

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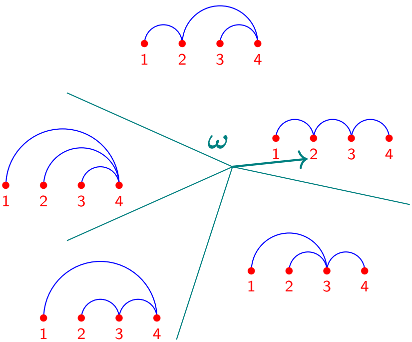

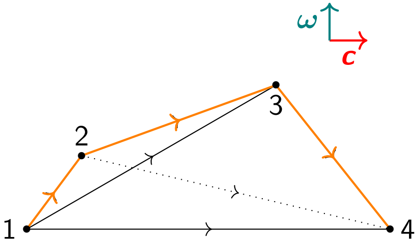

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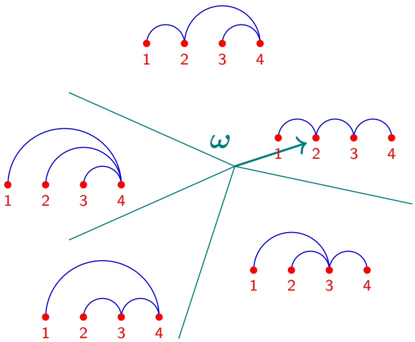

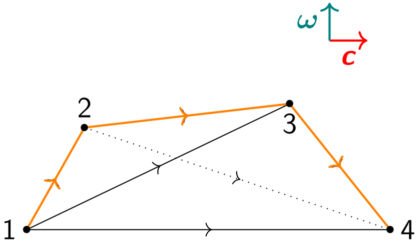

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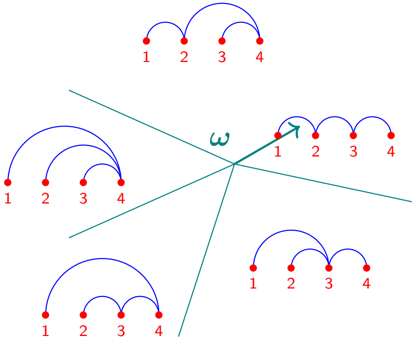

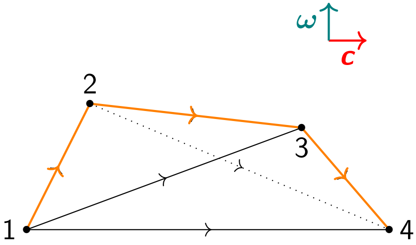

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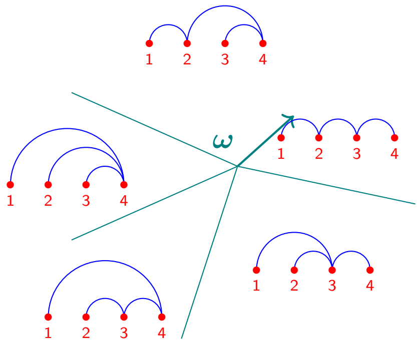

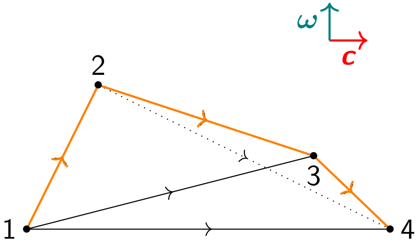

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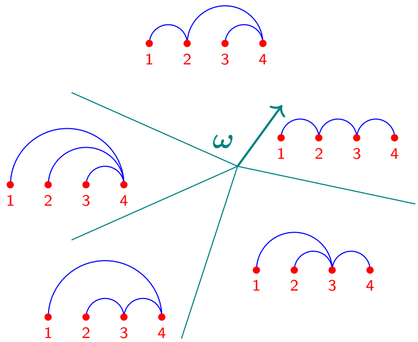

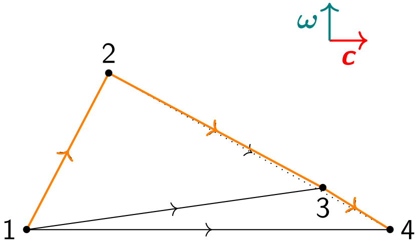

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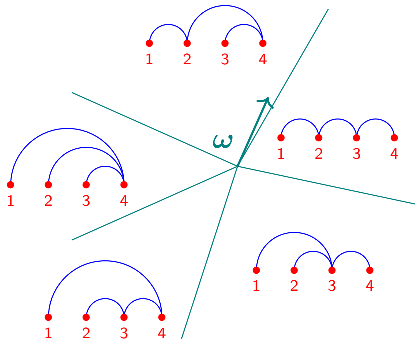

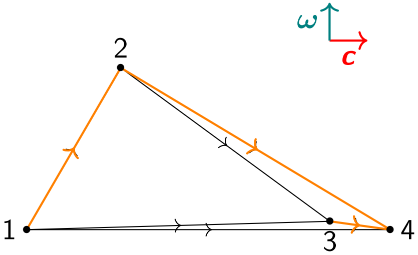

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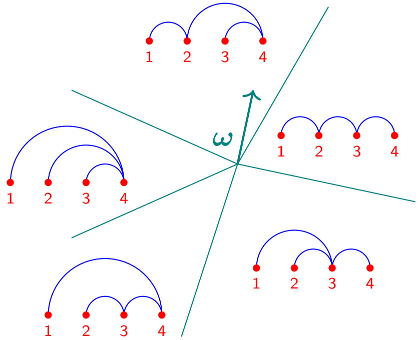

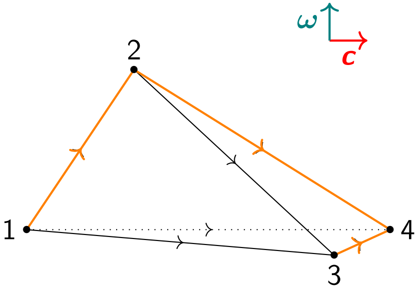

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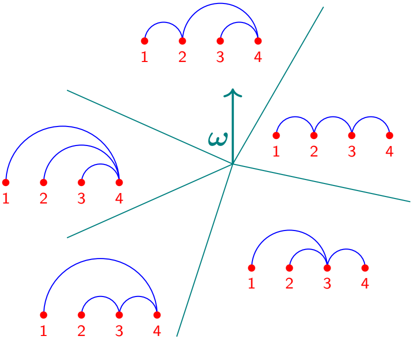
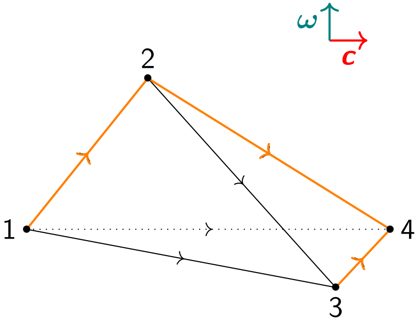
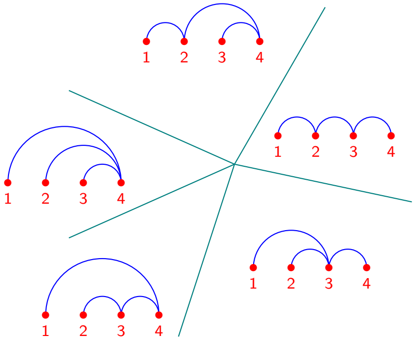




(Max-slope) pivot polytope



Coherent arborescence:
arborescence obtained via max-slope pivot rule

Pivot rule fan:
$\omega \sim \omega'$ iff same arborescence.

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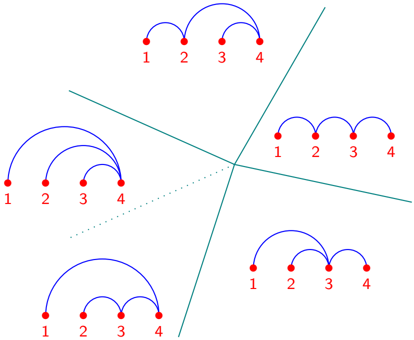

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Pivot rule fan refines monotone path fan

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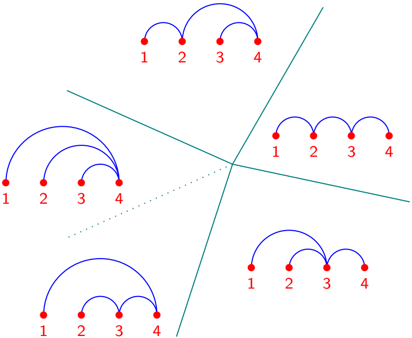

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The pivot rule fan is polytopal.
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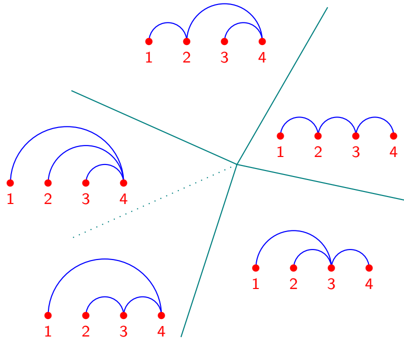

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$$\Sigma_{\boldsymbol{c}}(\Delta_d) = \mathrm{Cube}_{d-1} \qquad \Pi_{\boldsymbol{c}}(\Delta_d) = \mathrm{Asso}_d$$

Link with generalized permutahedra?

Theorem (Black, De Loera, Lütjeharms, Sanyal '23+)

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$\Pi_{\boldsymbol{c}}(\Delta_{d_1} \times \Delta_{d_2}) \simeq \mathrm{Asso}_{d_1} \star \mathrm{Asso}_{d_2}$ $\star$ shuffle product

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Challenge 1: Prove the conjecture!

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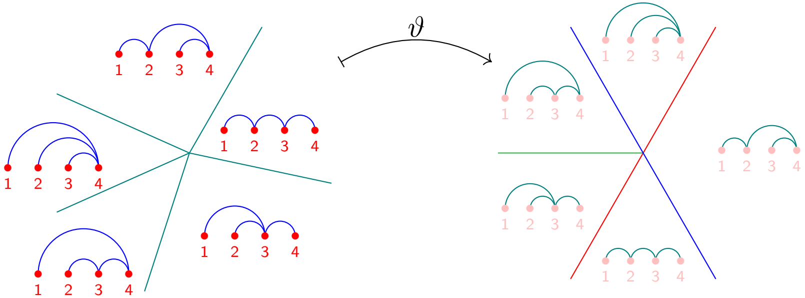
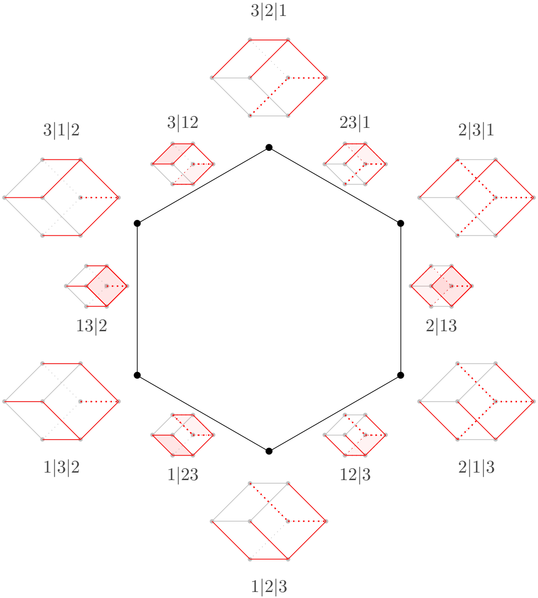
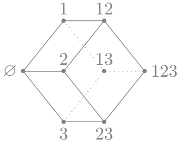
Challenge 1: Prove the conjecture!
Challenge 2: Give **geometric** proofs!

Case of the d-simplex



Case of the d-cube

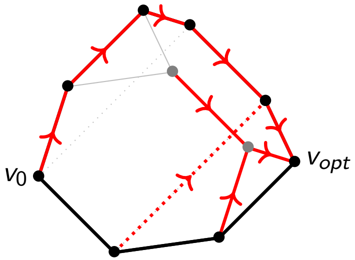




Slope comparisons

Max-slope pivot rule: take (improving) neighbor with best slope

For ω, what is important?



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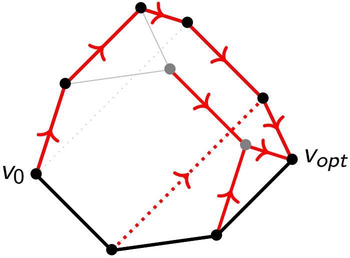

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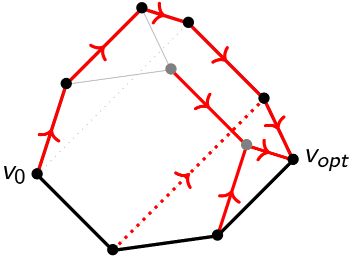

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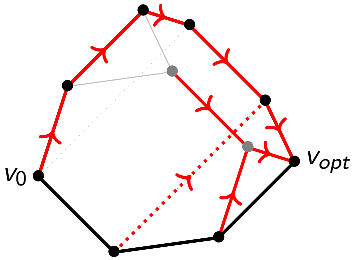
$\theta : \mathbb{R}^d \to \mathbb{R}^m$, injective linear map



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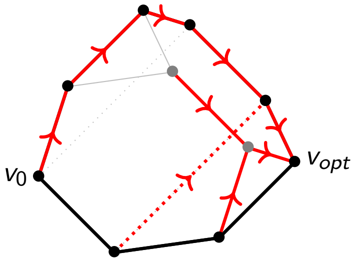
What is **really** important??



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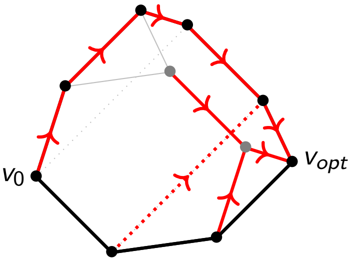

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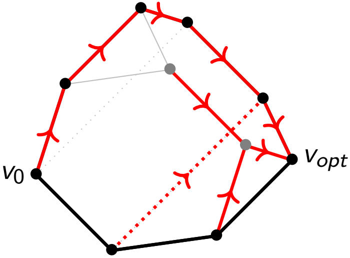
What is **really** important?? The comparisons of slopes!

Max-slope pivot rule: take (improving) neighbor with best slope



For $\boldsymbol{\omega}$, what is important?

Slopes: $\tau_{\boldsymbol{\omega}}(u, v) = \frac{\langle \boldsymbol{\omega}, u-v \rangle}{\langle \boldsymbol{c}, u-v \rangle}$

Slope vector:
$\theta(\boldsymbol{\omega}) = \big(\tau_{\boldsymbol{\omega}}(u, v)\big)_{uv \in E(\mathrm{P})}$

$\theta : \mathbb{R}^d \to \mathbb{R}^m$, injective linear map

What is **really** important?? The comparisons of slopes!

Compare coordinates of $\theta(\boldsymbol{\omega})$

Max-slope pivot rule: take (improving) neighbor with best slope

For ω, what is important?

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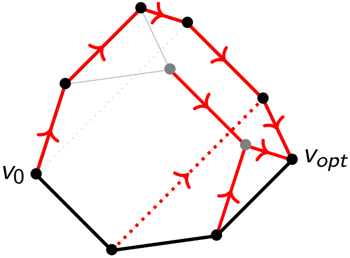


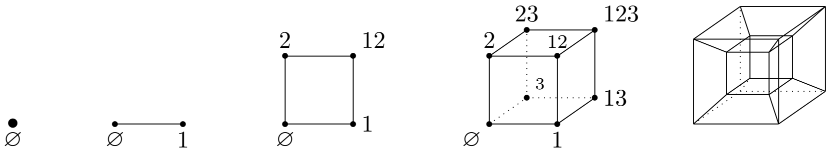
$\theta : \mathbb{R}^d \to \mathbb{R}^m$, injective linear map

What is **really** important?? The comparisons of slopes!

Compare coordinates of $\theta(\omega)$

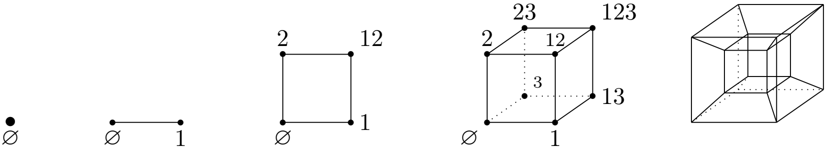
Where is $\theta(\omega)$ in the braid fan $\mathcal{B}_m$?

Case of the d-cube

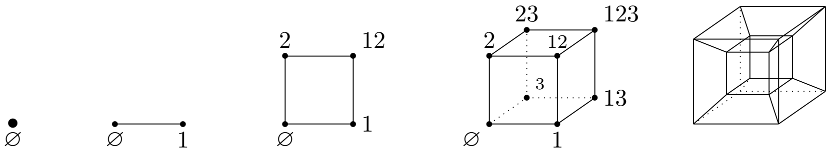


Too many edges

Case of the d-cube

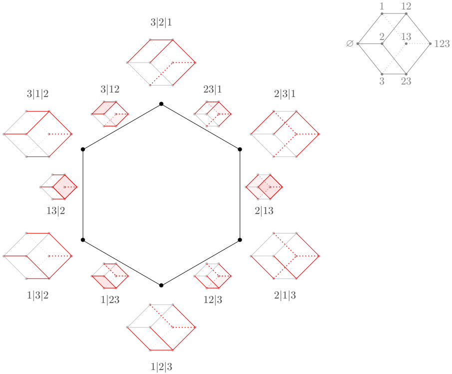


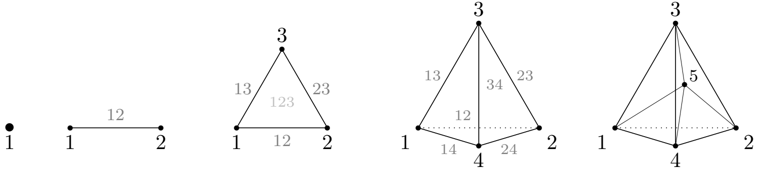
Too many edges, **but** parallelism saves us!



Too many edges, **but** parallelism saves us!

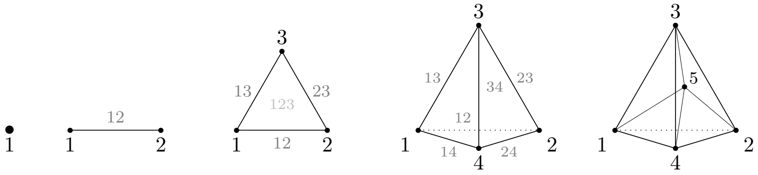
Geometric proof of $\Pi_{\boldsymbol{c}}(\mathrm{Cube}_d) = \Pi_d$



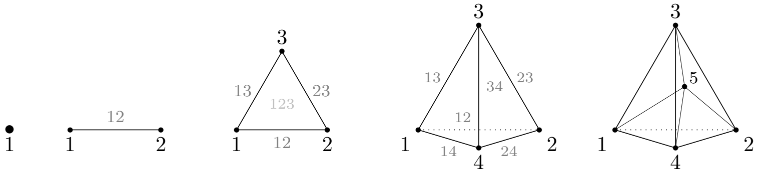


Too many edges

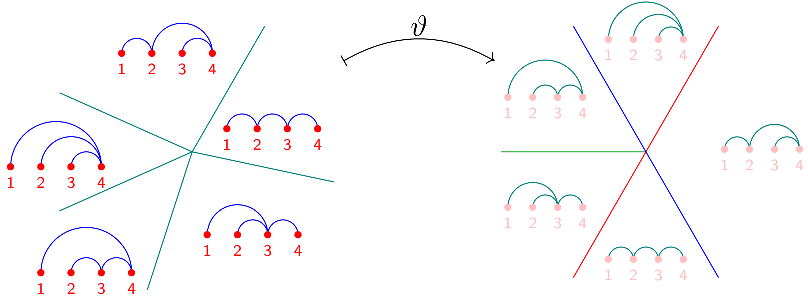
Case of the d-simplex



Too many edges, **but** affine independence saves us!



Too many edges, **but** affine independence saves us!
Geometric proof of $\Pi_{\boldsymbol{c}}(\Delta_d) = \mathrm{Asso}_d$



Shuffles

Shuffle: $(E, \leq)$ and $(F, \preceq)$ posets, then $\trianglelefteq$ is a shuffle when:

groud set :	$E \sqcup F$
relations :	all relations of $\leq$; all relations of $\preceq$;
	for each $e \in E$, $f \in F$, choose if $e \trianglelefteq f$ or $e \trianglerighteq f$
	$(+$ transitive closure)

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Theorem (Chapoton, Pilaud '22)

P, Q*: generalized permutahedra.*
Exists polytope $\mathsf{P} \star \mathsf{Q}$ *s.t.*

$$\mathcal{P}(\mathsf{P} \star \mathsf{Q}) = \{\textit{all shuffles between } \leq \in \mathcal{P}(\mathsf{P}) \textit{ and } \preceq \in \mathcal{P}(\mathsf{Q})\}$$

Product of simplices

Combine parallelism & affine independence:

Theorem

For $\Delta_{d_1} \times \cdots \times \Delta_{d_r}$, all (generic) direction:

$$\Pi_{\boldsymbol{c}}(\Delta_{d_1} \times \cdots \times \Delta_{d_r}) \simeq \mathrm{Asso}_{d_1} \star \cdots \star \mathrm{Asso}_{d_r}$$

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Example

(a) $\Pi_{\boldsymbol{c}}(\square_d) \simeq \mathrm{Perm}_d$
(b) $\Pi_{\boldsymbol{c}}(\square_m \times \Delta_n) \simeq (m, n)$-multiplihedron
(c) $\Pi_{\boldsymbol{c}}(\Delta_m \times \Delta_n) \simeq (m, n)$-constrainahedron

Ongoing work

1) For which P, $\Pi_{\boldsymbol{c}}(\mathrm{P})$ is a generalized permutahedron?
$\longrightarrow$ a priori, only products of simplices, but no proof

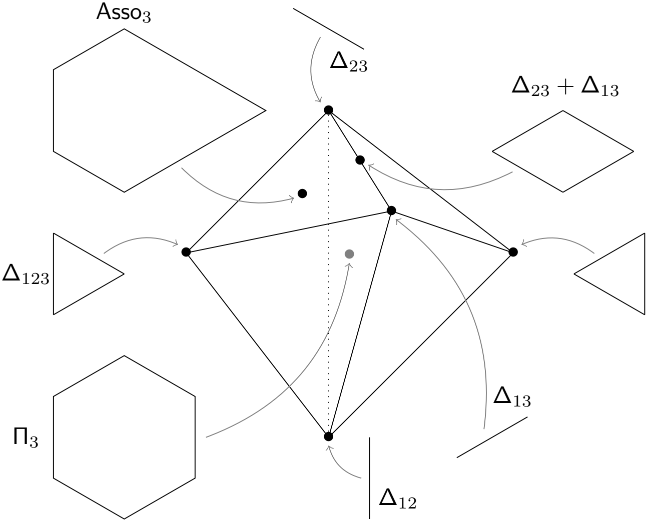
2) Is $\Pi_{\boldsymbol{c}}(\mathrm{P})$ projection of a generalized permutahedron?
$\longrightarrow$ pivot fan sent inside $\mathrm{Im}(\theta) \cap \mathcal{B}_m$

3) When $\Pi_{\boldsymbol{c}}(\mathrm{P})$ and $\Pi_{\boldsymbol{c}}(\mathrm{Q})$ **not** generalized permutahedra, what happen to $\Pi_{\boldsymbol{c}}(\mathrm{P} \times \mathrm{Q})$?
$\longrightarrow$ not equivalent to $\Pi_{\boldsymbol{c}}(\mathrm{P}) \star \Pi_{\boldsymbol{c}}(\mathrm{Q})$, but "embeds" in it

What I have presented

Contents

Thank you!



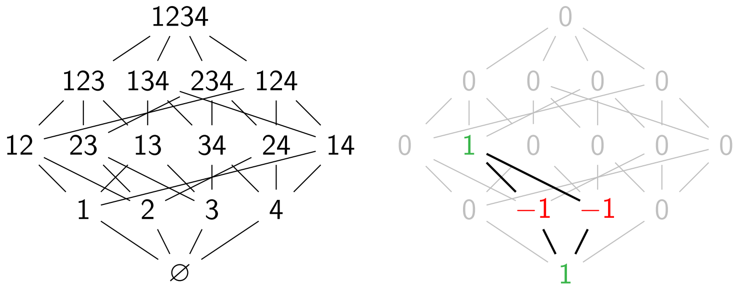
Asso_3
Δ_{23}
$\Delta_{23} + \Delta_{13}$
Δ_{123}
Δ_{13}
Π_3
Δ_{12}

Bonus slides...

Notations: $Sx = S \cup \{x\}$, $(\boldsymbol{f}_X)_{X \subseteq [n]}$ canonical basis of $\mathbb{R}^{2^{[n]}}$

Definition

Submodular vector $\boldsymbol{n}(S, u, v) = \boldsymbol{f}_{Suv} - \boldsymbol{f}_{Su} - \boldsymbol{f}_{Sv} + \boldsymbol{f}_S$
for $u, v \in S \subseteq [n]$



The tool: submodular dependancies

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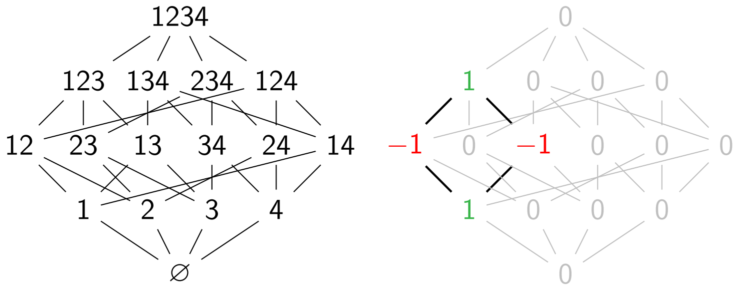


The tool: submodular dependancies

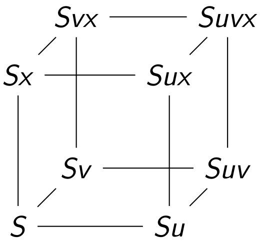
Lemma (Submodular normal)

$\boldsymbol{n}(S, u, v)$ are the facet's normals of $\mathbb{DC}(\Pi_n)$

Lemma (Cubic relation)

$u, v, x \notin S \subseteq [n]$

$$\boldsymbol{n}(Suvx, u, v) + \boldsymbol{n}(Sux, u, x) = \boldsymbol{n}(Suv, u, v) + \boldsymbol{n}(Suvx, u, x)$$



NB: Cubic relations generates all relations of submodular vectors

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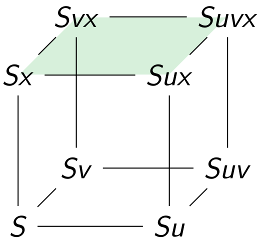

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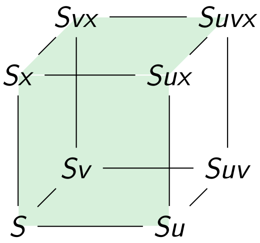

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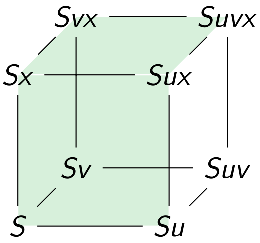
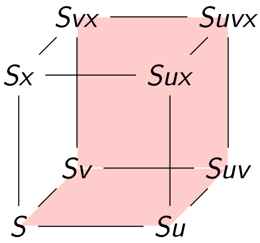




NB: Cubic relations generates all relations of submodular vectors

Monotone path polytope and pivot rule polytope

Let $\mathrm{P} \subset \mathbb{R}^d$ be a polytope.
Max-slope pivot rule: $A^{\boldsymbol{\omega}}(v) = \operatorname{argmax}\left\{\frac{\langle \boldsymbol{\omega}, u-v\rangle}{\langle \boldsymbol{c}, u-v\rangle} ; u \text{ impr. neig. of } v\right\}$.

Coherent monotone path: A monotone path that can be obtained via the max-slope pivot rule.
Monotone path polytope $\Sigma_{\boldsymbol{c}}(\mathrm{P})$ [**?**]: Fiber polytope of $\mathrm{P} \xrightarrow{\pi} \mathrm{Q}$ with Q a segment. (Can be seen as a Minkowski sum of sections of P.)
The vertices of $\Sigma_{\boldsymbol{c}}(\mathrm{P})$ are all coherent monotone paths.

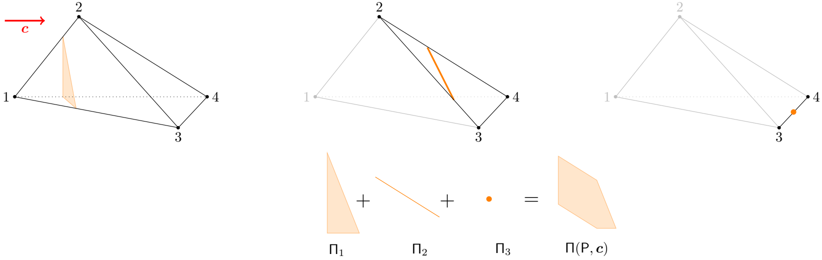
Coherent arborescence: An arborescence that can be obtained via the max-slope pivot rule.
Pivot rule polytope $\Pi_{\boldsymbol{c}}(\mathrm{P})$: Polytope which vertices are all coherent arborescences.

$$\Pi_{\boldsymbol{c}}(\mathrm{P}) = \operatorname{conv}\left\{\sum_{v \neq v_{opt}} \frac{1}{\langle \boldsymbol{c}, A(v) - v\rangle}(A(v) - v) ; A \text{ coherent arbo. of P}\right\}$$

Monotone path polytope and pivot rule polytope

Coherent arborescence: An arborescence that can be obtained via the max-slope pivot rule.
Pivot rule polytope $\Pi_{\boldsymbol{c}}(\mathsf{P})$: Polytope which vertices are all coherent arborescences. Can also be seen as a Minkowski sum of sections:

$$\sum_{v \in V(\mathsf{P})} (\text{section between } v \text{ and its improving neighbors})$$



Mimicking the case of the d-cube

Idea 1:

Fix a polytope P, and direction $\boldsymbol{c}$, n vertices, m edges.

$\theta : \mathbb{R}^d \to \mathbb{R}^m$ sends the pivot fan inside $\mathrm{Im}(\theta) \cap \mathcal{B}_m$

Problem: This is not a braid fan as $d << m$...

If m' classes of parallelism:

$\overline{\theta} : \mathbb{R}^d \to \mathbb{R}^{m'}$ sends the pivot fan inside $\mathrm{Im}(\theta) \cap \mathcal{B}_{m'}$

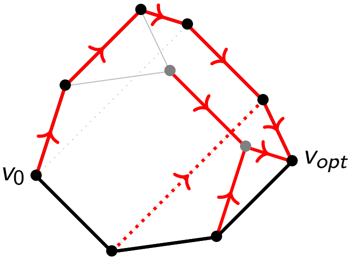
Problem: This is not a braid fan as $d << m' < m$...

We need to go lower dimensional!

Adapted slope map

Idea 2:
Fix a polytope P, direction $\boldsymbol{c}$,
n vertices, m edges.

Fix A arborescence:
$\vartheta_A(\boldsymbol{\omega}) = \big(\tau_{\boldsymbol{\omega}}(u, A(u))\ ;\ u \text{ vertex}\big)$



ϑ_A: linear, injective, $\mathbb{R}^d \to \mathbb{R}^{n-1}$
but if $\boldsymbol{\omega}$ does not capture A, then $\vartheta_A(\boldsymbol{\omega})$ have no meaning...

Adapted slope map: $\vartheta(\boldsymbol{\omega}) = \vartheta_{A^{\boldsymbol{\omega}}}(\boldsymbol{\omega})$
i.e. take $\boldsymbol{\omega}$ and look at the slope of the edges it selects.

Case of the d-simplex

$d = n - 1 \iff$ P is a simplex
For Δ_d: $\vartheta : \mathbb{R}^d \to \mathbb{R}^d$ piece-wise linear, $\ker \vartheta = \{\mathbf{0}\} \Rightarrow$ bijection
ϑ sends the pivot fan of Δ_d inside $\mathcal{B}_d$.