Pivot rule polytopes of products of simplicies

Vincent Pilaud[1], **Germain Poullot**[2]

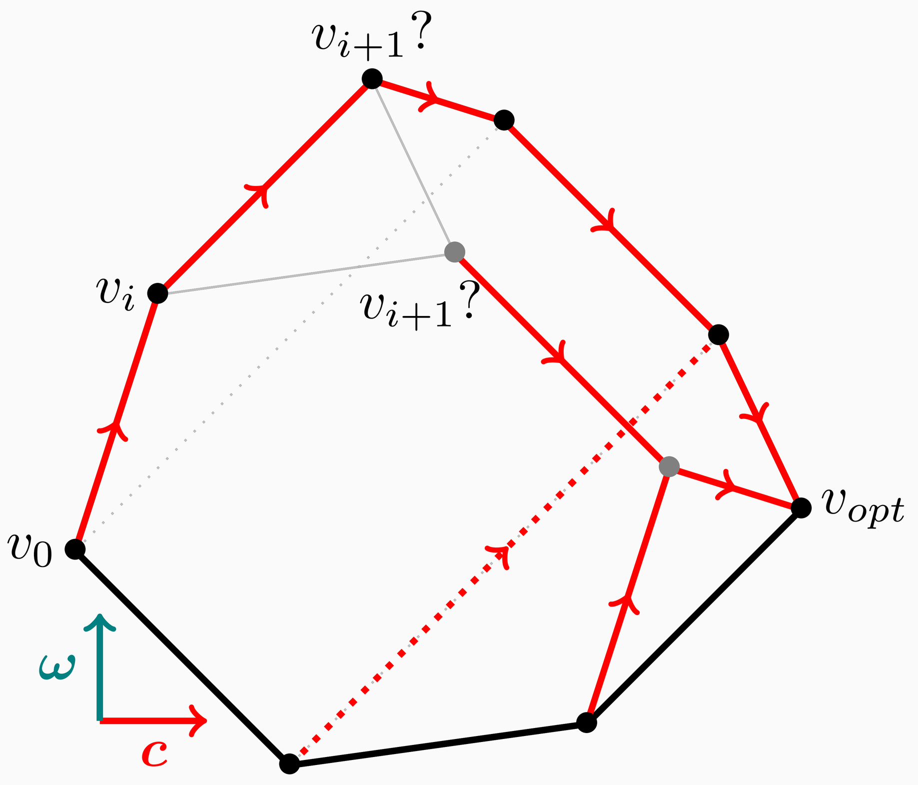
[1] Departament de Matemàtiques i Informàtica, Universitat de Barcelona & CRM, Spain vincent.pilaud@ub.edu
[2] Institut für Mathematik, Universität Osnabrück, Germany germain.poullot@uni-osnabrueck.de

arXiv:2410.12658

UNIVERSITAT DE BARCELONA UNIVERSITÄT OSNABRÜCK

Shadow vertex rule

Linear program $(\mathsf{P}, \boldsymbol{c})$: how to choose next vertex in simplex method?



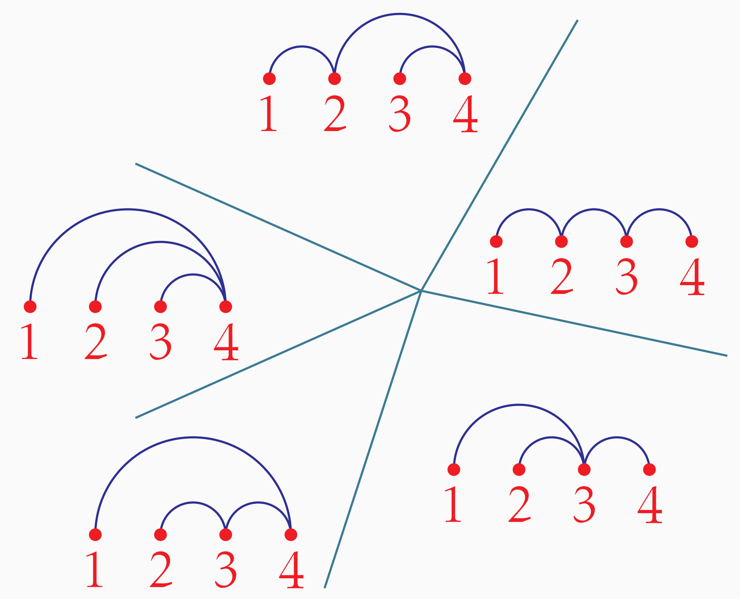
Shadow vertex: for ω, project in plane $(\boldsymbol{c}, \omega)$, take the neighbor with the best slope:

$$A^{\omega}(v) = \operatorname{argmax}\left\{\frac{\langle \omega, u - v\rangle}{\langle \boldsymbol{c}, u - v\rangle} \;;\; \begin{array}{l} u \text{ improving} \\ \text{neighbor of } v \end{array}\right\}$$

Pivot (rule) polytope

Coherent arborescence: monotone arborescence arising from shadow vertex rule

Pivot rule fan: $\omega \sim \omega'$ iff same arborescence



Pivot rule polytope $\Pi_c(\mathsf{P})$: dual to pivot rule fan

$\mathrm{Vert}\big(\Pi_c(\mathsf{P})\big) \longleftrightarrow \boldsymbol{c}$-coherent arborescences

Resemble Billera–Sturmfels' fiber polytopes

Loday Asso_n

Loday associahedron

$$\mathsf{Asso}_n = \sum_{i<j} \operatorname{conv}\{e_k \;;\; i \le k \le j\}$$

$\mathrm{Vert}\big(\mathsf{Asso}_n\big) \longleftrightarrow$ non-crossing partitions
$\longleftrightarrow$ binary trees,...

Pivot polytopes of Δ_n, Cube_n, ...

THM. [BLLS23] & [BS24] for (almost) all $\boldsymbol{c}$,

$$\begin{aligned} \Pi_c(\Delta_{n+1}) &\simeq \mathsf{Asso}_n \\ \Pi_c(\mathsf{Cube}_n) &\simeq \mathsf{Permuto}_n \\ \Pi_c([0,1] \times \Delta_{n+1}) &\simeq \mathsf{Multiplihedron}_n \end{aligned}$$

CONJ. [PS23] for (almost) all $\boldsymbol{c}$,

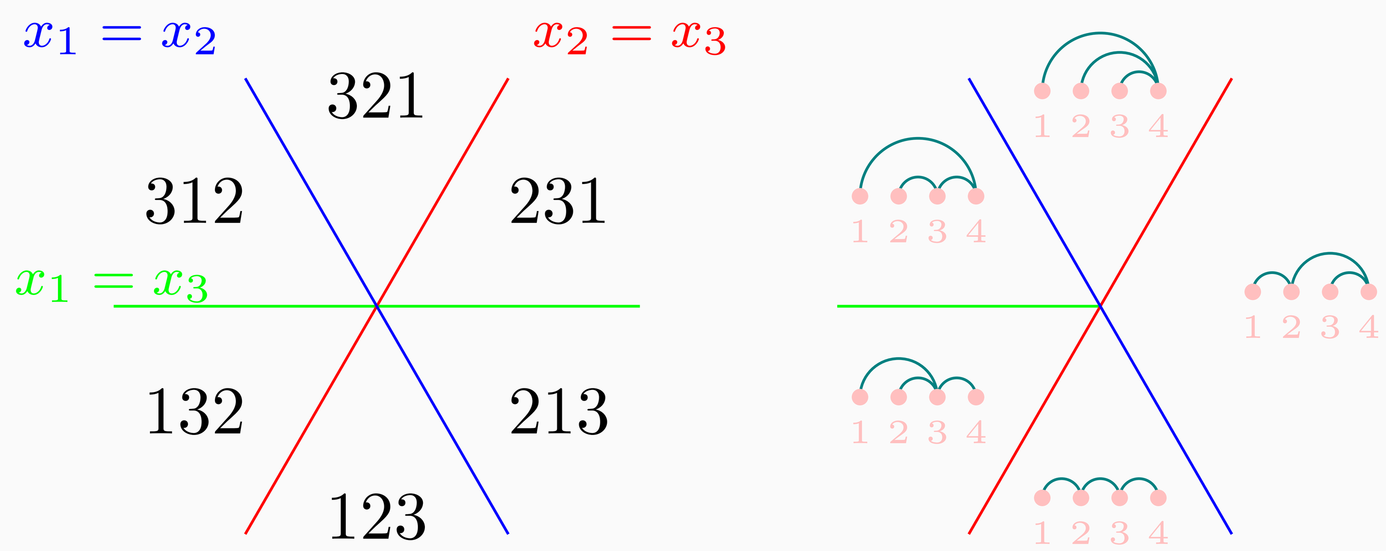
$\Pi_c(\Delta_{m+1} \times \Delta_{n+1}) \simeq \mathsf{Constrainahedron}_{m,n}$

Why!?

Braid fan

Braid fan: hyperplane arrangement $\{x \;;\; x_i = x_j\}$

Coarsening: choose maximal cones and merge



Shuffle

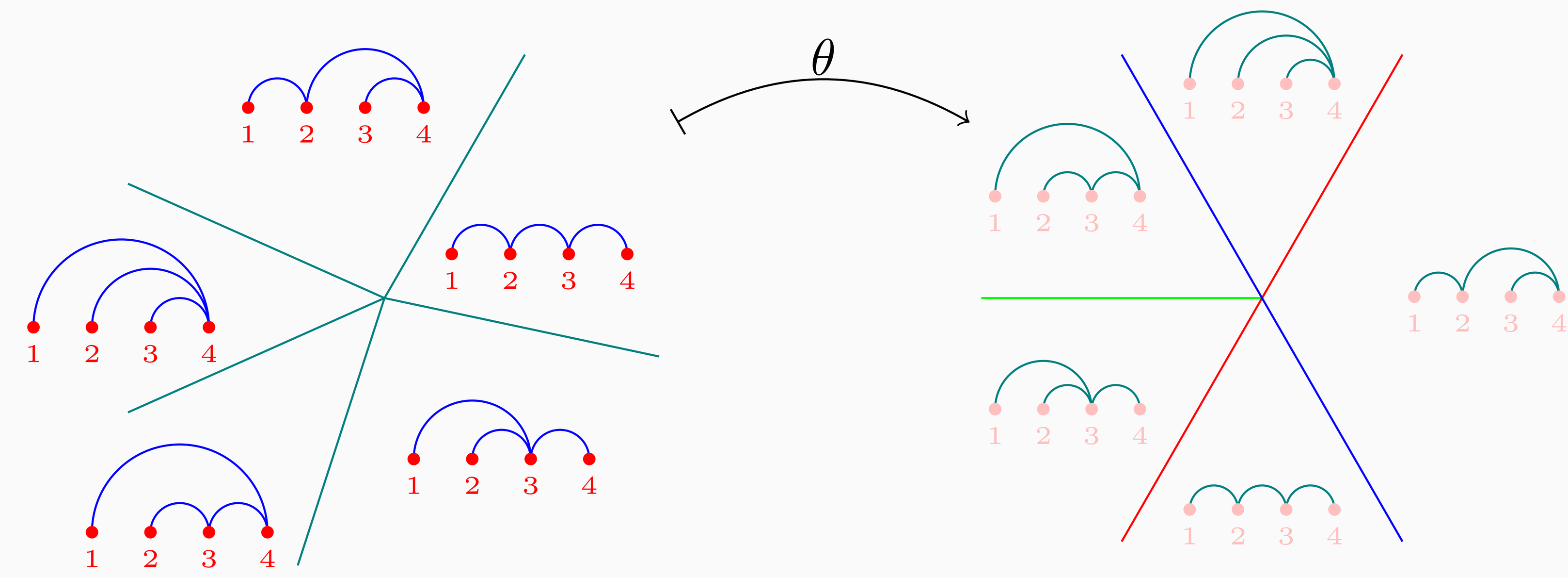
[CP24] For $\mathsf{P} \subseteq \mathbb{R}^m$, $\mathsf{Q} \subseteq \mathbb{R}^n$ deformed permutahedra, their *shuffle* is:

$$\mathsf{P} \star \mathsf{Q} = (\mathsf{P} \times \mathsf{Q}) \;+ \sum_{i\in[m],\, j\in[n]} \big[e_i, e_{m+j}\big] \qquad \subseteq \mathbb{R}^{m+n}$$

Slope map

For $\omega \in \mathbb{R}^d$, *slope* at vertex v is $\tau^{\omega}(v) = \max\left\{\frac{\langle \omega, u-v\rangle}{\langle c, u-v\rangle} \;;\; \begin{array}{l} u \text{ improving} \\ \text{neighbor of } v \end{array}\right\}$

Slope map $\theta : \mathbb{R}^d \to \mathbb{R}^{n-1}$, with $\theta(\omega) = \Big(\tau^{\omega}(v_1), \tau^{\omega}(v_2), \dots, \tau^{\omega}(v_{n-1})\Big)$



LEM. $G(\mathsf{P})$ complete $\Rightarrow A^{\omega}$ determined by **order of coordinates** of $\theta(\omega)$

THM. θ is a piecewise linear iso. from pivot fan $(\Delta_{n+1}, \boldsymbol{c})$ to n. fan Asso_n

Pivot polytopes of product of simplices

For $\mathsf{P} = \mathsf{P}_1 \times \cdots \times \mathsf{P}_r$ with slope maps $\theta_1, \dots, \theta_r$ the *product slope map* is $\Theta : \mathbb{R}^d \to \mathbb{R}^{n-1}$, with $\Theta(\omega_1, \dots, \omega_r) = \Big(\theta_1(\omega_1), \dots, \theta_r(\omega_r)\Big)$

THM. Θ is a piecewise linear map from pivot fan of $(\Delta_{d_1+1} \times \cdots \times \Delta_{d_r+1}, \boldsymbol{c})$ to normal fan $\mathsf{Asso}_{d_1} \star \cdots \star \mathsf{Asso}_{d_r}$.