Indecomposable polytopes & Rays of the submodular cone

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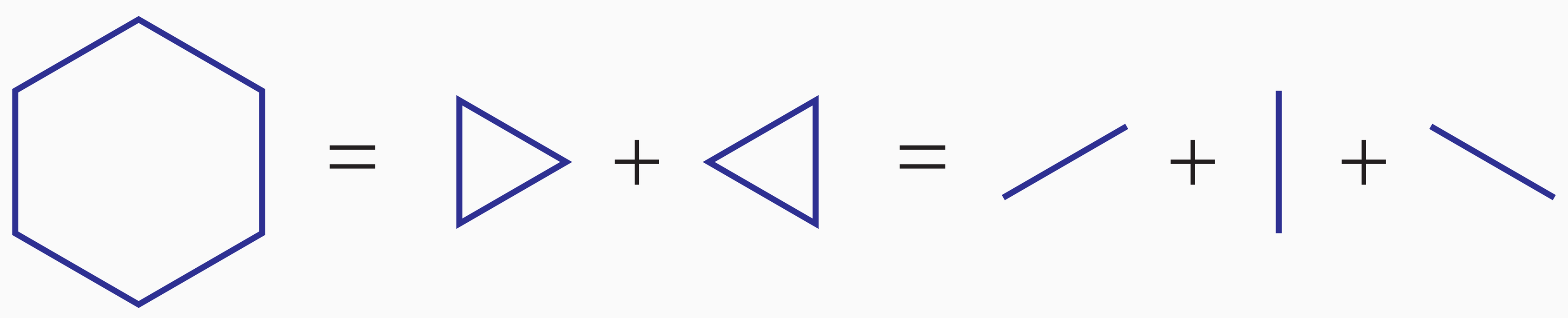
(In)decomposability of polytopes

Minkowski sum: $\mathsf{Q}+\mathsf{R}=\{q+r\ ;\ q\in\mathsf{Q}, r\in\mathsf{R}\}$

Indecomposable: If $\mathsf{P}=\mathsf{Q}+\mathsf{R}$, then $\mathsf{Q}=\lambda\mathsf{P}+t$ for some $\lambda>0$, t



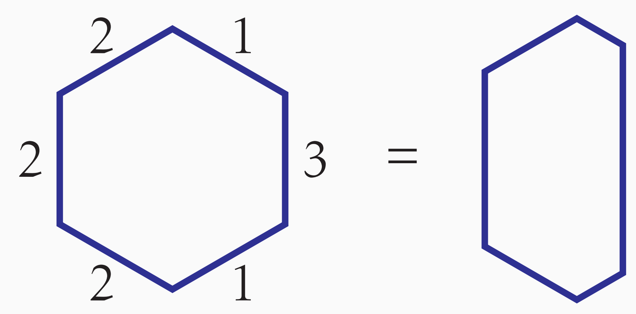
Question A: How to check if P is indecomposable?

Parameterizing deformations of P

Deformations of P: any Q s.t. $\lambda\mathsf{P}=\mathsf{Q}+\mathsf{R}$ for some $\lambda>0$ and R

$\longleftrightarrow$ edges of Q parallel to edges of P, but not same length

$\longleftrightarrow$ *edge-length vector*: λ_{e} for each edge e of P, $\lambda_{\mathsf{e}}(\mathsf{Q})=\frac{\text{length}(\mathsf{e}\in\mathsf{Q})}{\text{length}(\mathsf{e}\in\mathsf{P})}$

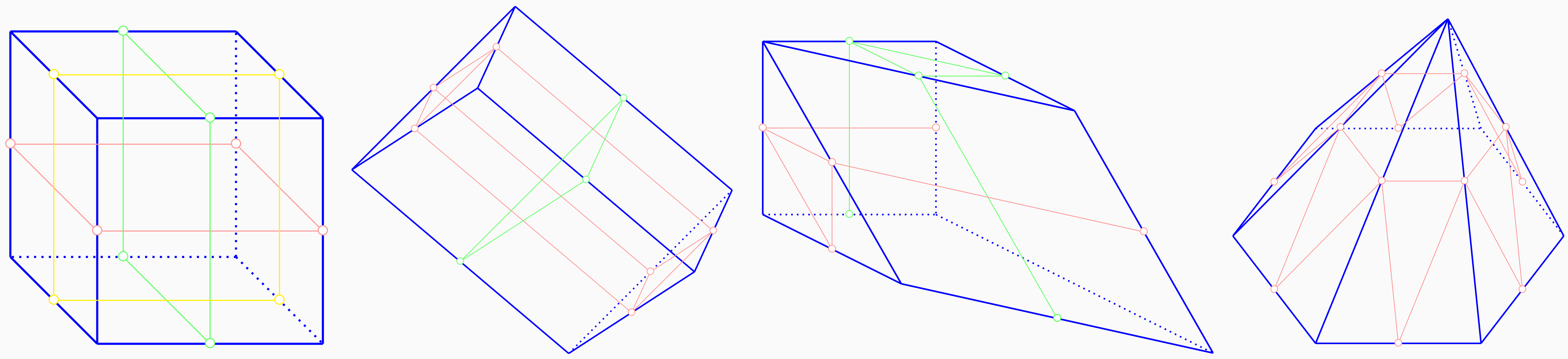


Cycle eqn.: if cycle $\mathsf{e}_1,\dots,\mathsf{e}_r$, then for all Q: $\lambda_{\mathsf{e}_1}(\mathsf{Q})\vec{e}_1+\dots+\lambda_{\mathsf{e}_r}(\mathsf{Q})\vec{e}_r=0$

Edge-length dependencies graph: new criteria for indecomposability

Dependent edges e, f: for all deformations Q of P, $\lambda_{\mathsf{e}}(\mathsf{Q})=\lambda_{\mathsf{f}}(\mathsf{Q})$ informally: "the length of e can be deduced from the length of f"

Examples of dependent edges: edges of a triangle; edges of a cycle whose convex hull is a simplex; edges opposite in a parallelogram or a trapezoid...



Graph of edge-length dependencies $ED(\mathsf{P})$: nodes = edges of P; arcs ef if e and f dependent edges

THM. $ED(\mathsf{P})$ is a clique $\Longleftrightarrow$ $ED(\mathsf{P})$ is connected $\Longleftrightarrow$ P is indecomposable $\dim\mathbb{DC}(\mathsf{P})\leq|cc(ED(\mathsf{P}))|$

THM. $\exists$ subset S of dependent edges, forming a connected sub-graph of the graph of P, such that S touches every facet $\Longrightarrow$ P indecomposable

Application: Re-deduce almost all previous indecomposability criteria **Application:** Study decomposability products of polytopes $\mathsf{P}\times\mathsf{P}'$

Edmonds's problem

Deformed permutahedron: all edge directions e_i-e_j

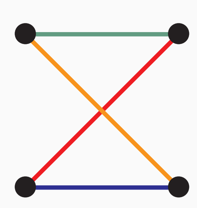
Edmonds '70 problem: Find (all) indecomposable deformed permutahedra!

Nguyen '86: give $2^{2^{n-3/2\log n+O(1)}}$ indecomposable deform. permutahedra via connected matroids

Question B: How to create indecomposable deformed permutahedra, not matroid polytopes?

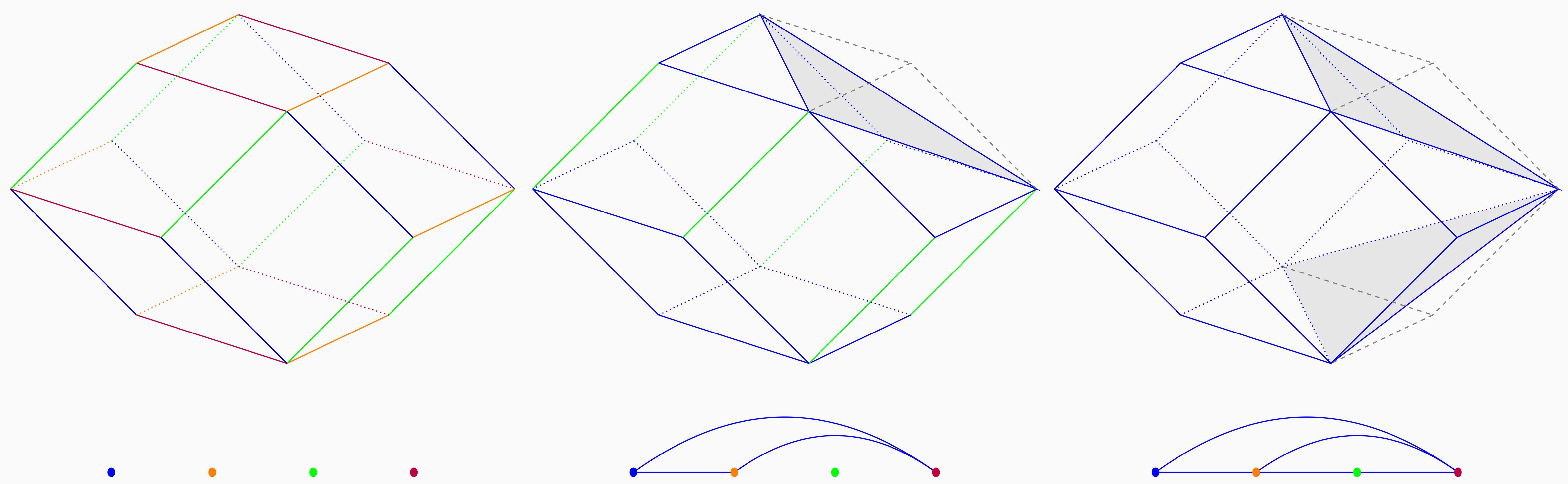
Idea: use *graphical zonotope* of graph G:

$$\mathsf{Z}_G:=\sum_{ij\in E}[e_i,e_j]\qquad K_{2,2}=$$



Vertices Z_G $\longleftrightarrow$ acyclic orientations of G

Truncating zonotopes



Application: Pick a zonotope Z + remove vertices = Z' $\rightarrow$ easy to tell if Z' is indecomposable

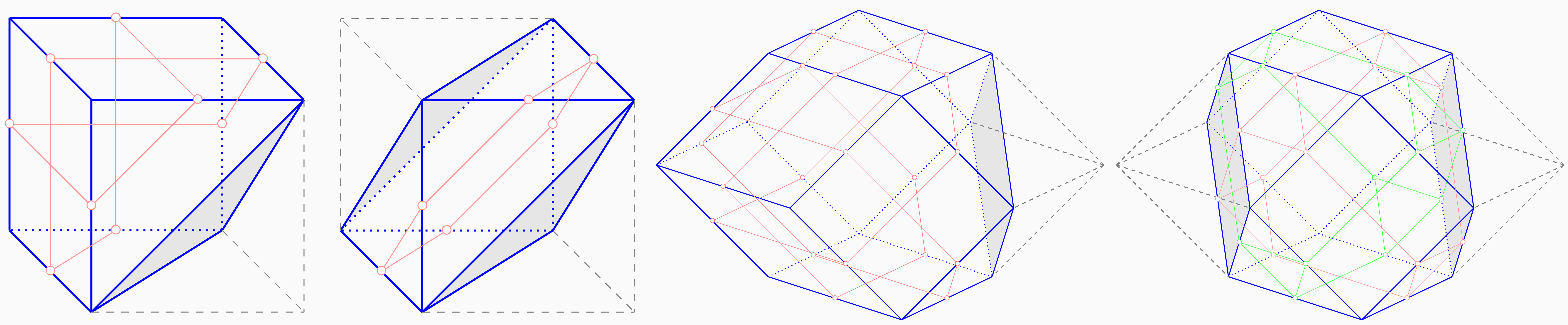
Idem for stacking vertices

New rays of the submodular cone: truncated graphical zonotopes of complete bipartite graphs

Works *only* when $G=K_{n,m}$ complete bipartite graph

$\mathsf{Z}^{-\circ}_{n,m}$, $\mathsf{Z}^{\circ-\circ}_{n,m}$: graphical zonotope $\mathsf{Z}_{K_{n,m}}$, remove 1 or 2 specific vertices

$$\mathsf{Z}^{-\circ}_{n,m}:=\mathsf{Z}_{K_{n,m}}\cap\left\{x\in\mathbb{R}^{n+m}\ ;\ \sum_{j=1}^{m}x_{b_j}\leq nm-1\right\}\quad\text{and}\quad\mathsf{Z}^{\circ-\circ}_{n,m}:=\mathsf{Z}^{-\circ}_{n,m}\cap\left\{x\in\mathbb{R}^{n+m}\ ;\ \sum_{i=1}^{n}x_{a_i}\leq nm-1\right\}$$



THM. $\mathsf{Z}^{-\circ}_{n,m}$ and $\mathsf{Z}^{\circ-\circ}_{n,m}$ are indecomposable deformed $(n+m)$-permutahedra, not matroid polytope except 5 cases

THM. There are $\geq 2^{\left\lfloor\frac{d-1}{2}\right\rfloor}$ such indecomposable deformed d-permutahedra Soon on ArXiv: $\geq 2^{2^n}$ (Loho–Padrol–Poullot, other method)