Many rays of the Submodular Cone

Georg Loho[1], Arnau Padrol[2], **Germain Poullot**[3]

[1] Freie Universität Berlin, Germany & University of Twente, Netherlands — georg.loho@math.fu-berlin.de
[2] Departament de Matemàtiques i Informàtica, Universitat de Barcelona & CRM, Spain — arnau.padrol@ub.edu
[3] Institut für Mathematik, Universität Osnabrück, Germany — germain.poullot@uni-osnabrueck.de

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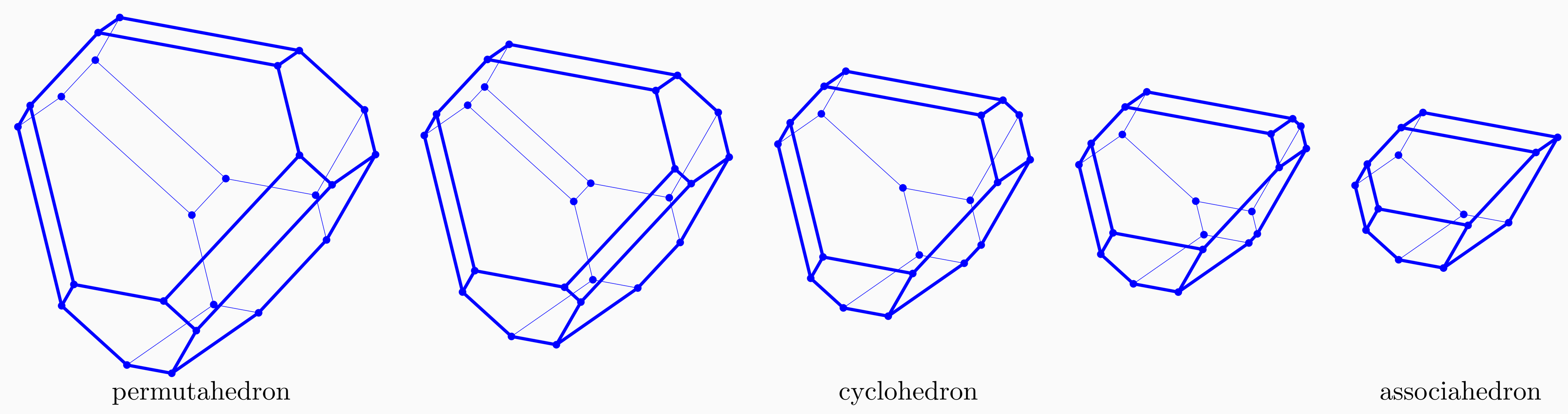
Deformation cone



Q is a *deformation* of P if $\lambda P = Q + R$ for some $\lambda > 0$ and some R

Deformation cone $\mathbb{DC}(P) := \{Q \,;\, Q \text{ is deformation of } P\}$

$\mathbb{DC}(P)$ is closed by Minkowski sums and positive dilation: $\mathbb{DC}(P)$ is a cone.

Submodular cone $\mathbb{SC}_n$

Permutahedron: $\mathrm{conv}\Big\{\big(\sigma(1),\dots,\sigma(n)\big) \,;\, \sigma\in\mathcal{S}_n\Big\}$

Submodular cone: $\mathbb{SC}_n := \mathbb{DC}(\mathsf{Permuto}_n)$

= cone submodular functions $f : 2^{[n]} \to \mathbb{R}$:

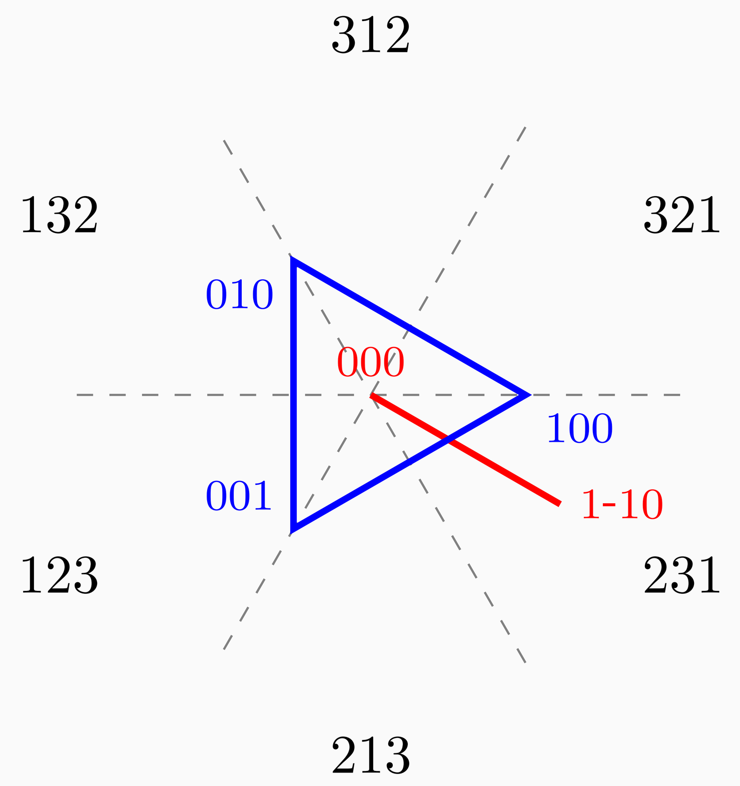
$$f(A) + f(B) \geq f(A\cap B) + f(A\cup B)$$

Edmonds '70 problem: Find (all) rays of $\mathbb{SC}_n$ *a.k.a.* all "indecomposable deformed permuta."

Nguyen '86: gives $2^{2^{n-1/2\log n + O(1)}}$ rays via connected matroids (= rays on 0/1-coordinates)

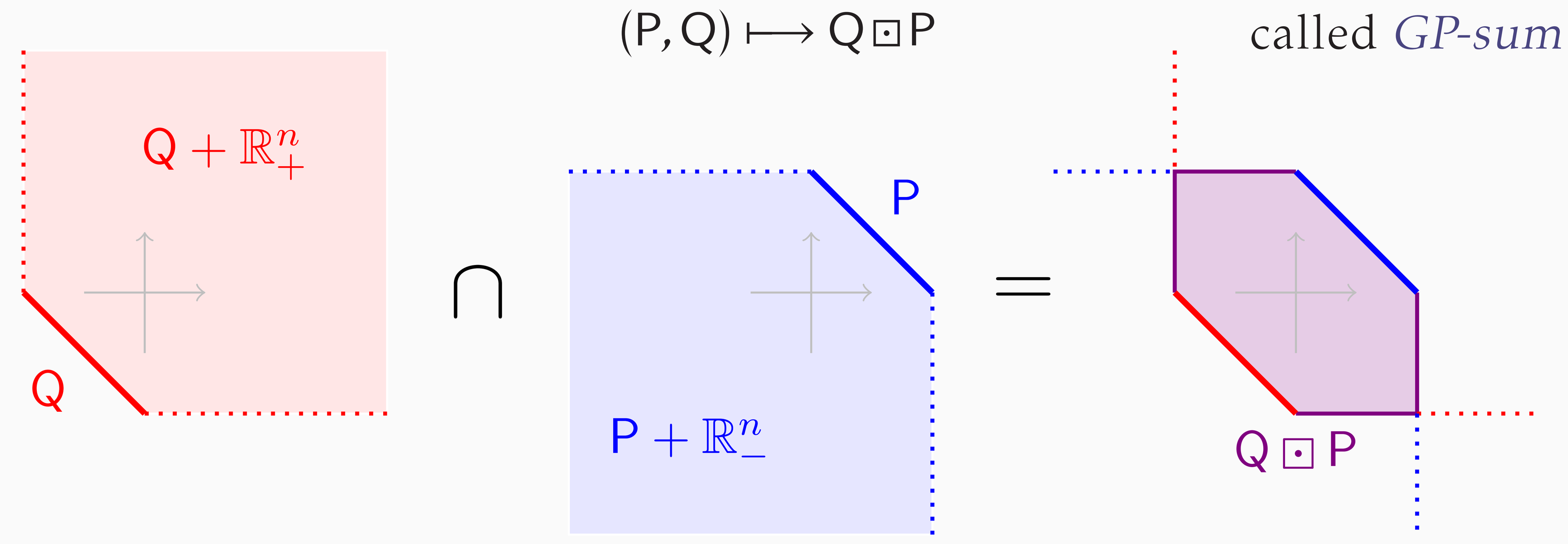
GP-sum

$(P,Q)\in\mathbb{SC}_n\times\mathbb{SC}_n$ is *admissible* if $\forall\sigma\in\mathcal{S}_n,\, i\in[n],\ (P^\sigma)_i \leq (Q^\sigma)_i$

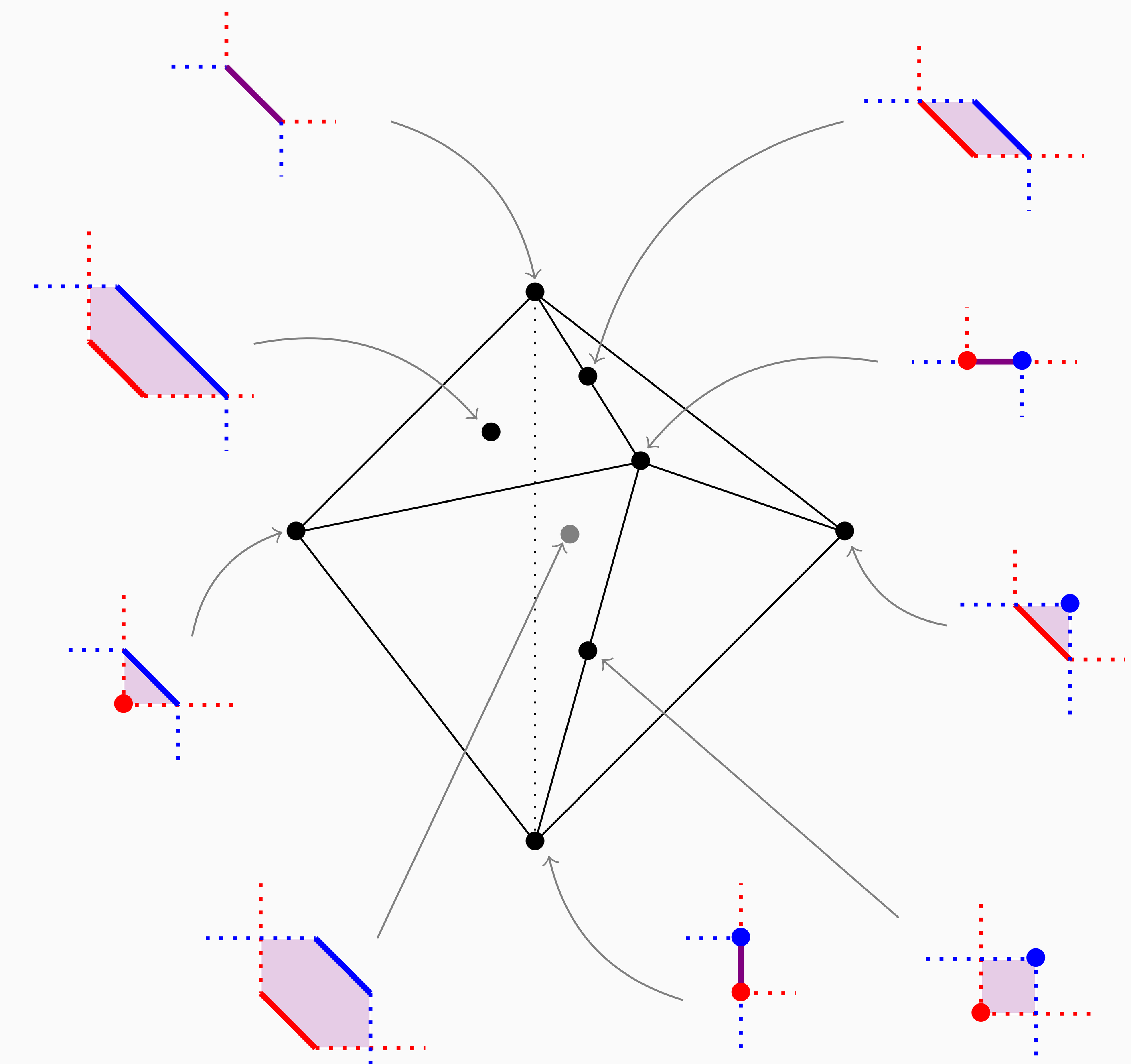


"Bijection" between {admissible pairs $\in\mathbb{SC}_n\times\mathbb{SC}_n$} and $\mathbb{SC}_{n+1}$:

$(P,Q)\longmapsto Q\boxdot P$ called *GP-sum*



$\mathbb{SC}_3 = \mathcal{C}(\text{segment, segment})$



$\mathcal{C}(P,Q) = \Big\{Q'\boxdot P' \,;\, P',Q' \text{ deformations } P,Q \text{ and admissible pair}\Big\}$

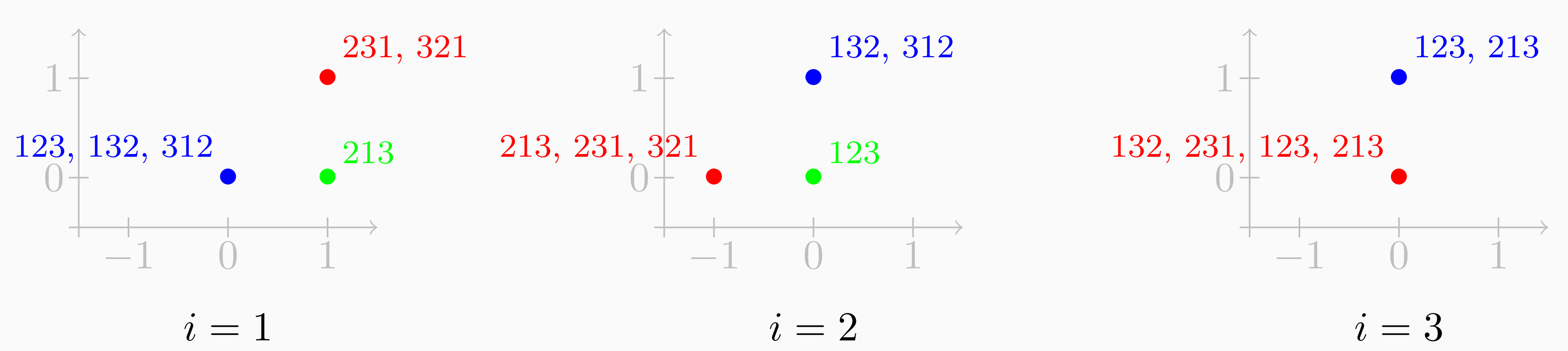
THM. $\mathcal{C}(P,Q)$ is a deformation cone

Seeds and fertility

Fix P, Q indecomposable, *i.e.* rays of $\mathbb{SC}_n$

Seed for (P, Q): some λ, t such that $(\lambda Q + t)\boxdot P$ indecomposable

LEM. (P, Q) has seed iff no top-left-most point in one of the pictures:



(P, Q) is *fertile* if (P, Q) and (Q, P) have seeds

LEM. P, Q, R, S pairwise fertile $\implies$ $(Q\boxdot P, R\boxdot S)$ fertile pair

p_n = number of pairwise fertile rays of $\mathbb{SC}_n$

THM. $\quad p_{n+1} \geq p_n^2$

Many rays

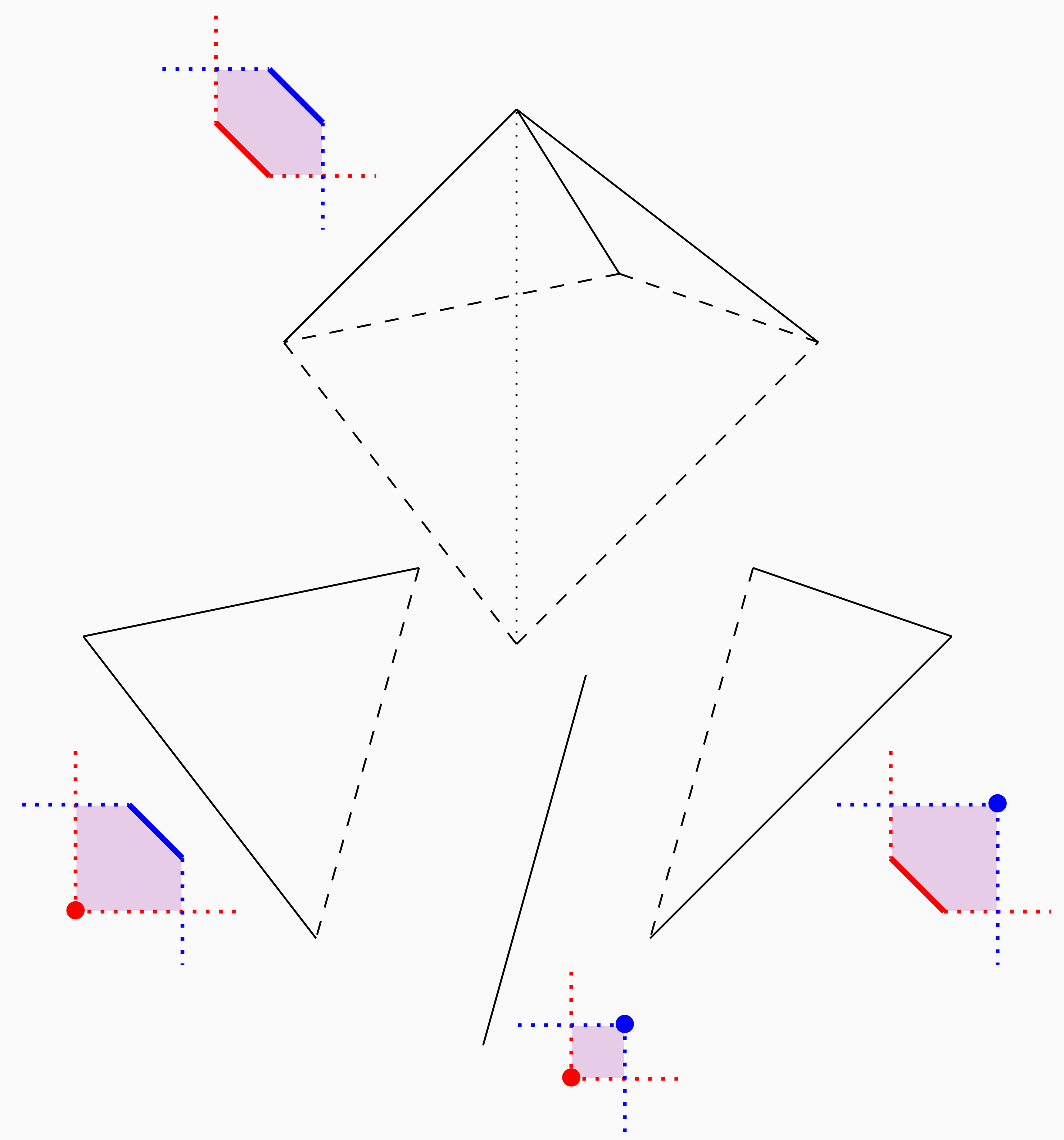
$$2^{2^{n-2}} \leq \#\text{ rays } \mathbb{SC}_n \leq n^{2^n}$$

PROOF:

rays $\mathbb{SC}_n \leq$ Upper Bound Theorem (may be improved?)

rays $\mathbb{SC}_n \geq p_n$ with $p_{n+1}\geq p_n^2$ and $p_5 \geq 656$

Going further



Our method provides partition of $\mathbb{SC}_{n+1}$ parameterized by $\mathbb{SC}_n\times\mathbb{SC}_n$.