Discrete Mathematics - Übungsblatt

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This exercise sheet contains some exercises to help you understand the course. They are more or less in the order in which the notions are presented in the lecture.

Exercises marked with ∘ are exercises to help you understand the definitions of the course.

Exercises marked with ⋆ are exercises you can choose to present (on the board). Every week, one of you will present an exercise on the board (mandatory for having the right to attend to the exam).

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Simplicial Complexes – Summer Semester 2025

Repetitorium 1

Exercise 1 [Counting simplicial complexes ∘]

Show that there are $2^{\binom{n}{d+1}} - 1$ pure d-dimensional simplicial complexes on n vertices (NB: the empty complex is not pure of dimension d).

Exercise 2 [Alexander dual ⋆]

Let Δ be a simplicial complex on the vertex set $[n]$. We define

$$\Delta^* = \{F \subseteq [n] \ : \ [n] \setminus F \notin \Delta\}.$$

We call Δ^* the *dual* simplicial complex or the *Alexander dual* of Δ. Show that:

(a) Δ^* is a simplicial complex.
(b) $(\Delta^*)^* = \Delta$.
(c) The number of facets of Δ^* equals the number of minimal non-faces of Δ.
(d) Prove that $f_k(\Delta) + f_{n-k-1}(\Delta^*) = \binom{n}{k-1}$.
(e) [With the theorem of principal decomposition] Suppose that the Stanley-Reisner ideal of Δ is generated as follows: $I_\Delta = \langle \boldsymbol{x}^{a_1}, \ldots, \boldsymbol{x}^{a_s} \rangle$ with the usual notation $\boldsymbol{x}^{a_i} = \prod_{j=1}^n x_j^{a_{i,j}}$ with $a_{i,j} \in \{0,1\}$. Show that:

$$I_{\Delta^*} = \bigcap_{i=1}^{s} \langle x_j \ ; \ \text{for } j \in [n], \ \text{if } a_{i,j} = 1 \rangle$$

Exercise 3 [Join ⋆]

Let Δ and Γ be simplicial complexes on disjoint vertex sets V and W. The *join* $\Delta * \Gamma$ is the simplicial complex on the vertex set $V \cup W$ with faces $F \cup G$, where $F \in \Delta$ and $G \in \Gamma$. Express the h-vector $h(\Delta * \Gamma)$ in terms of $h(\Delta)$ and $h(\Gamma)$.

Exercise 4 [Operations on ideals ∘]

Let $I, J \subseteq k[x_1, \ldots, x_n]$ be monomial ideals. Show that:

(a) $I \cap J$ is a monomial ideal generated by

$$G(I \cap J) = \{\operatorname{lcm}(f, g) \ : \ f \in G(I), \ g \in G(J)\}$$

(b) $I + J$ is a monomial ideal and $G(I + J) \subseteq G(I) \cup G(J)$.
(c) $I \cdot J$ is a monomial ideal and $G(I \cdot J) \subseteq G(I)G(J)$.

Exercise 5 [Order complex of a poset ⋆]

Let $P = (V, \leq)$ be a partially ordered set (i.e., $a \leq a$ for $a \in V$; $a \leq b$ and $b \leq a$ imply $a = b$; and $a \leq b$ and $b \leq c$ imply $a \leq c$). The order complex of P is the simplicial complex $\Delta(P)$ of all subsets $\{v_1, \ldots, v_i\} \subseteq V$, $i \geq 0$, such that (for a suitable choice of indices) $v_1 < v_2 < \ldots < v_i$. Show that:

(a) $\Delta(P)$ is a simplicial complex on the vertex set V.
(b) Determine the minimal non-faces of $\Delta(P)$ and the Stanley-Reisner ideal $I_{\Delta(P)}$.
(c) Compute $\Delta(P)$ and $I_{\Delta(P)}$ for some partially ordered sets of your choice.

Exercise 6 [r-skeleton ∘]
Let Δ be a $(d-1)$-dimensional simplicial complex and $\Delta^{\langle r\rangle} = \{F \in \Delta \,:\, \dim F \leq r\}$ the r-skeleton of Δ for $0 \leq r \leq d-1$. Compute the f- and h-vector of $\Delta^{\langle r\rangle}$ using the f- and h-vector of Δ.

Exercise 7 [Alternating complex ⋆]
Let $\Omega_n = \{-1, \dots, -n, 1, \dots, n\}$. Let Γ_n be the set of all subsets $F \subseteq \Omega_n$ such that $|(\{-i, i\} \cap F)| \leq 1$ for $1 \leq i \leq n$.
(a) Show that Γ_n is a simplicial complex.
(b) Compute the f-vector of Γ_n.

Exercise 8 [h_i are integers ∘]
Show that $h_i \in \mathbb{Z}$, for all f-vector and all i.

Exercise 9 [h-vectors ∘]
Compute the h-vectors corresponding to these f-vectors using Stanley's trick:
(a) $f(\Delta) = (1, 6, 12, 8)$,
(b) $f(\Delta) = (1, 12, 66, 108, 54)$,
(c) $f(\Delta) = (1, 3, 72, 118, 59)$.
(d) the f-vector of (the boundary complex of) a simplex
(e) the f-vector of (the boundary complex of) a bi-pyramid over a triangle
(f) the f-vector of (the boundary complex of) an octahedron

Exercise 10 [h-vector of r-skeleton ∘]
Fix a simplicial complex Δ. Draw its Pascal-like triangle to compute its h-vector. Show that the r^{th} row of this triangle is the h-vector of the r-skeleton $\Delta^{\langle r\rangle}$.

Exercise 11 [Multiplication in polynomial ring ∘]
Fix $f \in \mathbb{F}[x_1, \dots, x_n]$, and consider the application $\text{mult}_f : \mathbb{F}[x_1, \dots, x_n] \to \mathbb{F}[x_1, \dots, x_n]$, defined by $\text{mult}_f(g) = f \cdot g$. Show that mult_f is a linear function, and write it in matrix form (in the monomial basis) Start with the case of f a monomial.

Exercise 12 [Examples of Stanley-Reisner rings ∘]
For the $(d-1)$-dimensional simplex Δ_{d-1}, what's $\mathbb{F}[\Delta_{d-1}]$?
For δ_n the simplicial complex whose facets are all singletons of $[n]$, what's $\mathbb{F}[\delta_n]$?

Exercise 13 [Truncated boundary of a simplex ⋆]
Let $m, n \in \mathbb{N}$ with $1 \leq m \leq n$, $F_i = \{1, \dots, n\} \setminus \{i\}$ for $i = 1, \dots, m$, and $\Delta = \langle F_1, \dots, F_m\rangle$ the abstract simplicial complex generated by $F_1, \dots, F_m$.
1. Show that $\mathbb{F}[\Delta] = \mathbb{F}[x_1, \dots, x_n]/(x_1 \cdots x_m)$.
2. Determine the h-vector of Δ.

Exercise 14 [A non-zero-divisor in the Stanley-Reisner ring ⋆]
Let Δ be a simplicial complex on vertex set $[n]$, with $f_0(\Delta) = n$. Let $\mathbb{F}[\Delta]$ be its Stanley-Reisner ring with coefficients.

1. Prove that the sum of all vertex variables $\theta := x_1 + x_2 + \ + x_n$ is always a non-zero-divisor in $\mathbb{F}[\Delta]$.
2. Prove that if Δ is disconnected, then the quotient ring $\mathbb{F}[\Delta]/(\theta)$ contains a non-zero element of degree 1 annihilated by all elements of positive degree. Conclude that every element of positive degree is a zero-divisor.

Exercise 15 [Minimal primary decomposition of I_Δ]
Show that the presentation od Satz 1.17 is irredundant (using the bijection between Stanley-Reisner ideals and simplicial complexes and Satz 1.17 itself).

Exercise 16 [Size of the basis of the $\mathbb{N}$-grading ∘]
Consider the standard $\mathbb{N}^n$-grading on $S = \mathbb{F}[x_1, \dots, x_n]$. What's $\dim S_a$?

Consider the standard $\mathbb{N}$-grading on $S = \mathbb{F}[x_1, \dots, x_n]$. Show that

$$\dim_{\mathbb{F}} S_i = \binom{n+i-1}{n-1} \text{ for } i \in \mathbb{N}$$

Exercise 17 [Hilbert series ∘]
Let $S = \mathbb{F}[X, Y, Z]$ be the standard graded polynomial ring in three variables over $\mathbb{F}$. Determine the Hilbert series of the following $\mathbb{F}$-algebras:

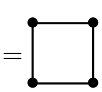
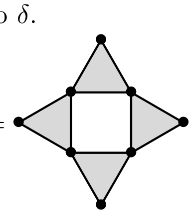
(a) $\mathbb{F}[X, Y, Z]/(XY^3)$,
(b) $\mathbb{F}[X, Y, Z]/(XY, XZ, Z^2)$,
(c) $\mathbb{F}[X, Y, Z]/(XY, XZ)$.

Exercise 18 [Hilber series (return) ∘]
Is $H(t) = \frac{1+2t-t^2}{(1-t)^2}$ the Hilbert series of a Stanley-Reisner ring?

Exercise 19 [Shellability of 1-dimensional simplicial complexes ∘]

1. Prove that a pure 1-dimensional simplicial complex is shellable if and only if it is a connected graph.
2. Prove that a pure 1-dimensional simplicial complex is partitionable if and only if it is a graph with no isolated vertices and at most one connected component which is a tree.

Exercise 20 [Shellability and deformation retract ∘]
Show that the complex δ is shellable but the complex Δ is not. Show that there exists a deformation retract sending Δ to δ.

$\delta =$  $\Delta =$ 

Exercise 21 [Shelling order $\circ$]
Find a shelling order for Belspiel 1.24.(iii).

Exercise 22 [Definitions of shellability $\star$]
Show the equivalence of the various definitions of shellability (cf. Definition 5.1).

Exercise 23 [Shellability for star and links $\circ$]
If Δ is shellable, show that $\text{star}_\Delta(F)$ and $\text{link}_\Delta(F)$ are shellable for all $F \in \Delta$.

Exercise 24 [Stanley-Reisner ring of the complete graph $\star$]
Let $\Delta := \{\emptyset, 1, 2, 3, 4, 12, 13, 14, 23, 24, 34\}$, the 1-dimensional simplicial complex which is the complete graph on 4 vertices.

1. Prove that for a field $\mathbb{F}$, the ring $\mathbb{F}[\Delta]$ contains a linear system of parameters (that is, θ_1, θ_2 in $\mathbb{F}[\Delta]$ of degree 1, such that $\mathbb{F}[\Delta]$ is finitely generated as a $\mathbb{F}[\theta_1, \theta_2]$-module) if and only if $\#\mathbb{F} \geq 3$, that is, $\mathbb{F} \neq \mathbb{F}_2$.
2. Show that the ring $\mathbb{Z}[\Delta]$ contains no pair of degree 1 elements $\theta_1, \theta_2 \in \mathbb{Z}[\Delta]$ for which $\mathbb{Z}[\theta_1, \theta_2]$ is finitely generated over $\mathbb{Z}[\theta_1, \theta_2]$.

Exercise 25 [Kind-Kleinschmidt criterion for an l.s.o.p. $\star$]
Let Δ be a $(d-1)$-dimensional simplicial complex on $[n]$. The restriction of a linear form $\theta = \sum_{i=1}^n \alpha_i x_i$ to a face $F \in \Delta$ is defined as $\theta\,|_F = \sum_{i \in F} \alpha_i x_i$.

1. Prove that linear forms $\theta^{(1)}, \ldots, \theta^{(d)} \in \mathbb{F}[x_1, \ldots, x_n]$ form a l.s.o.p. (linear system of parameters) for $\mathbb{F}[\Delta]$ if and only if $\theta^{(1)}\,|_F, \ldots, \theta^{(d)}\,|_F$ span an $|F|$-dimensional vector space for every face $F \in \Delta$.
2. Use the criterion above to show that there is no l.s.o.p. for the boundary complex of a d-simplex with coefficients in $\{0, 1\}$.

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Simplicial Complexes – Summer Semester 2025

Repetitorium 2

Exercise 26 [Homology groups ∘]

Let Δ be the simplicial complex with facets $\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{5\}$. Determine the reduced homology groups $\widetilde{H}_i(\Delta;k)$ for $i \geq 0$.

Exercise 27 [Cohen-Macaulay-ness ∘]

Decide whether the following simplicial complexes are Cohen–Macaulay. (In each case, the set of facets is given.)

(i) $\mathcal{F}(\Delta) = \{\{1,2,3\},\{3,4\}\}$
(ii) $\mathcal{F}(\Delta) = \{\{1,2,3\},\{2,3,4\}\}$
(iii) $\mathcal{F}(\Delta) = \{\{1,2,3\},\{2,3,4\},\{1,2,4\},\{1,3,4\}\}$
(iv) $\mathcal{F}(\Delta) = \{\{1,2,3\},\{4,5,6\}\}$

Exercise 28 [0^{th} reduced homology ∘]

Let Δ be a simplicial complex, and G the underlying graph (i.e. $G = \Delta^{\langle 1\rangle}$ is the 1-skeleton of Δ). Let T be a spanning forest of G. For each edge f of T, consider $\boldsymbol{e}_f \in C_1$ and its image $\partial_1(\boldsymbol{e}_f) \in C_0$.

Show that the family of vectors $\big(\partial_1(\boldsymbol{e}_f)\ ;\ f \text{ edge of } T\big)$ is a basis of $\operatorname{Im}\partial_1$.

Deduce $\tilde{H}_0(\Delta) = \mathbb{F}^{c-1}$, with c the number of connected components of Δ.

Exercise 29 [Cohen-Macaulay-ness and operations ∘]

Suppose Δ is Cohen-Macaulay. Prove that the link $\mathrm{lk}_\Delta(F)$ is Cohen-Macaulay for any face $F \in \Delta$.

Are sub-complexes and stars of Cohen-Macaulay simplicial complexe also Cohen-Macaulay?

Exercise 30 [Cohen-Macaulay-ness for graphs ∘]

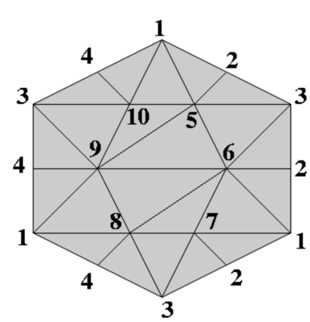
Which graphs are Cohen-Macaulay (use Reisner)? Compare with Exercise 19.

Exercise 31 [Cohen-Macaulay non-shellable ⋆]

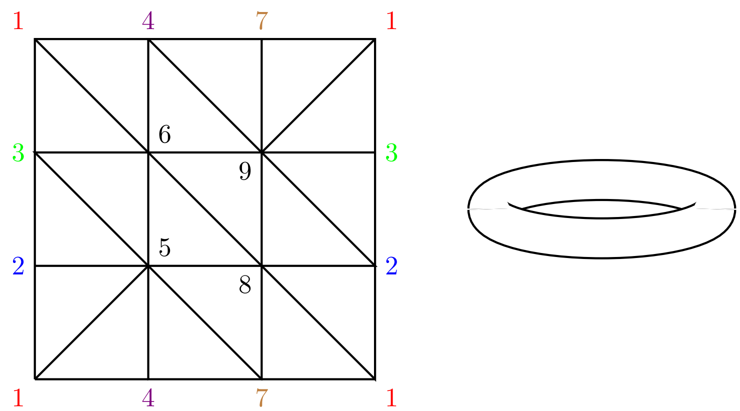
Prove that the following simplicial complex Δ is Cohen-Macaulay.

Prove that Δ is contractible but has no boundary: deduce that Δ is not shellable.

Compute its h-vector, and notice its positivity.



Exercise 32 [Triangulation of the torus ⋆]

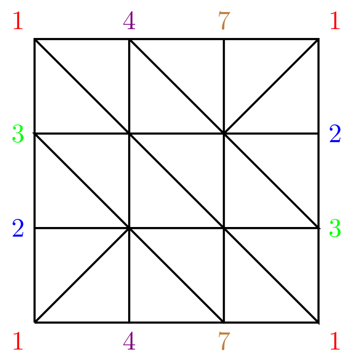


Torus $\mathbb{S}^1 \times \mathbb{S}^1$

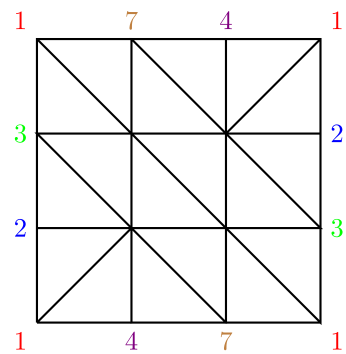
The above simplicial complex Δ is a triangulation of the torus $\mathbb{S}^1 \times \mathbb{S}^1$, all triangles are full (be careful, even if a vertex is repeated in the drawing, it is the **same** vertex in Δ).

0. Understand the drawing on the right side.
1. Compute the f-vector Δ.
2. Write the chain complex of Δ. Write $\operatorname{Ker}\partial_{-1}$, $\operatorname{Ker}\partial_0$, and $\operatorname{Im}\partial_0$.
3. Find a spanning tree of (the 1-skeleton of) Δ. Deduce $\operatorname{Ker}\partial_1$, and $\operatorname{Im}\partial_1$.
4. Looking at the chain complex, show that $\tilde{H}_1 = \mathbb{F}^{1+\dim \operatorname{Ker}\partial_2}$, and $\tilde{H}_2 = \mathbb{F}^{\dim \operatorname{Ker}\partial_2}$.
5. Prove that $\operatorname{Ker}\partial_2 = \mathbb{F}$. (Hint: you need to prove that there is exactly 1 linear relation between all $\boldsymbol{e}_t$ for t a triangle in Δ: pick a triangle, suppose that it has coefficient 1 in the linear relation, then deduce all other coefficients.)
6. Deduce all reduced homology groups. $\tilde{H}_1 = \mathbb{F}^2$, $\tilde{H}_2 = \mathbb{F}$
7. Conclude that a torus is not homeomorphic to a sphere nor a ball.
8. Is the torus Cohen-Macaulay?

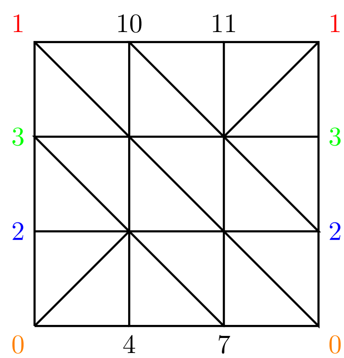
Exercise 33 [Triangulations of other surfaces ⋆]



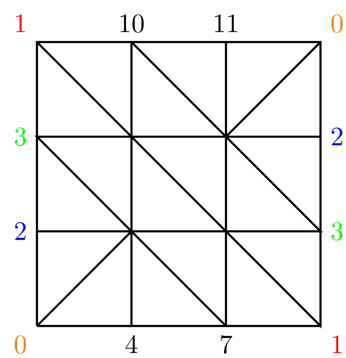
Klein bottle



Projective plane $P\mathbb{R}^2$

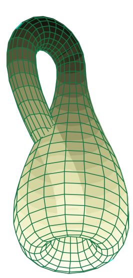


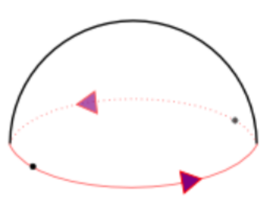
Cylinder

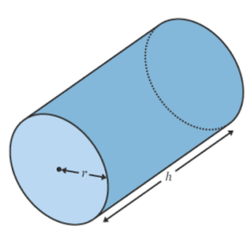


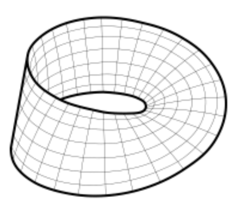
Möbius strip

These simplicial complexes are triangulations of the Klein bottle, the projective plane, the cylinder, and the Möbius strip: same questions as Exercise 31, and prove each they are not homeomorphic to one another, nor the torus, nor a ball. Which of the above triangulation correspond to which drawing?

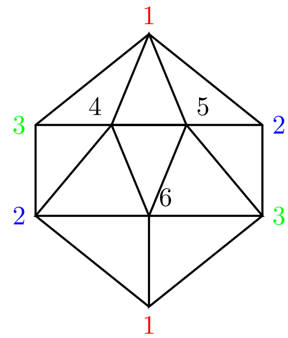








Exercise 34 [Triangulation of the projective plane (version 2) $\star$]



The above simplicial complex Δ is a triangulation of the projective plane, all triangles are full (be careful, even if a vertex is repeated in the drawing, it is the **same** vertex in Δ).

1. Compute the f-vector Δ.
2. Write the chain complex of Δ. Write $\operatorname{Ker}\partial_{-1}$, $\operatorname{Ker}\partial_0$, and $\operatorname{Im}\partial_0$.
3. Looking at the chain complex, show that $\tilde{H}_1 = \tilde{H}_2 = \mathbb{F}^{\dim \operatorname{Ker}\partial_2}$.
4. Find a spanning tree of (the 1-skeleton of) Δ. Deduce $\operatorname{Ker}\partial_1$, and $\operatorname{Im}\partial_1$.
5. Prove that $\operatorname{Ker}\partial_2 = \mathbf{0}$ if $1 \neq -1$ in $\mathbb{F}$, and $\operatorname{Ker}\partial_2 = \mathbb{F}$ if $1 = -1$ in $\mathbb{F}$ (e.g. $\mathbb{F} = \frac{\mathbb{Z}}{2\mathbb{Z}}$). (Hint: you need to prove that there is exactly 1 linear relation between all $\boldsymbol{e}_t$ for t a triangle in Δ: pick a triangle, suppose that it has coefficient 1 in the linear relation, then deduce all other coefficients.)
6. Deduce all reduced homology groups over all fields. $\tilde{H}_1 = \tilde{H}_2 = \mathbb{F}$ if $1 = -1$ in $\mathbb{F}$
7. Conclude that the projective plane is not homeomorphic to a sphere, nor a ball, nor the torus.
8. Is the projective plane Cohen-Macaulay?

Exercise 35 [Planar triangulations $\star$]

Consider a planar graph G, drawn in the plane, whose faces are triangles, except possibly the outer (infinite) face. Let Δ be the 2-dimensional simplicial complex whose 0-faces are the vertices of G, whose 1-faces are the edges of G, and whose 2-faces are all the aforementioned triangles.
Under which conditions (on G) is Δ Cohen-Macaulay? Shellable? Partitionable?

Exercise 36 [Feasible f-vectors $\circ$]
Decide whether the following vectors are f-vectors of simplicial complexes. If so, state such a complex.

a. $f = (1, 13, 69, 112, 56)$
b. $f = (1, 10, 40, 150)$

Exercise 37 [Regular lower shadows $\circ$]
(To checked that your are at easy with notations.)

Let $\mathcal{A} \subseteq 2^{[n]}$. We say that $\mathcal{A}$ is k-regular if $\mathcal{A} \subseteq \binom{[n]}{k}$. In general, we define the lower shadow of $\mathcal{A}$ to be $\Delta\mathcal{A} = \big\{B \in 2^{[n]} \ ; \ \exists A \in \mathcal{A}, x \in A,\ B = A \backslash \{x\}\big\}$.

1. Show that the above definition of shadows agrees with the one of the course.
2. Show that the shadow of $\mathcal{A}$ is $(k-1)$-regular if and only if $\mathcal{A}$ is k-regular.
3. Define $\Delta^{r+1}\mathcal{A} = \Delta(\Delta^r\mathcal{A})$ with $\Delta^0\mathcal{A} = \mathcal{A}$. What's $\#\Delta^m\mathcal{A}$ for $m = \max_{A\in\mathcal{A}} \#A$.

Exercise 38 [Without Kruskal–Katona $\circ$]
Let $\mathcal{A} \subseteq \binom{[n]}{k}$. **Without** using Kruskal–Katona theorem, show that:

$$\frac{k}{n-k+1}\#\mathcal{A} \ \le\ \#\Delta\mathcal{A} \ \le\ k\#\mathcal{A}$$

Compare with Kruskal–Katona theorem.

Exercise 39 [Squashed = co-lexicographic $\circ$]
Show that the following is an alternative definition of the squashed order:

For $A, B \in \binom{[n]}{k}$ with $A = \{a_1 > a_2 > \cdots > a_k\}$ and $B = \{b_1 > b_2 > \cdots > b_k\}$, we have $A \prec B$ if and only if there is $i \in [k]$ such that $a_i < b_i$ and $a_j = b_j$ for all $j < i$.

Exercise 40 [Squashed successor $\star$]
Let A be a k-element set of positive natural numbers. Show that: The successor of A in squashed order is: $B = [b] \cup \{a+1\} \cup (A\backslash[a])$, where a is minimal such that $a \in A$ and $a+1 \notin A$ and $b = \#(A \cap [a]) - 1$.

Exercise 41 [From Kruskal–Katona to Erdös–Ko–Rado $\circ$]
Use the Kruskal–Katona theorem to prove the Erdös–Ko–Rado theorem, i.e., prove the following statement: Let $1 \le i \le \frac{1}{2}n$ and let $\mathcal{A} \subseteq \binom{[n]}{i}$ such that for all $A, A' \in \mathcal{A}$ we have $A \cap A' \ne \emptyset$. Then $\#\mathcal{A} \le \binom{n-1}{i-1}$.

Exercise 42 [LYM inequality and Sperner theorem ⋆]
Sperner theorem state the following: "Let $\mathcal{A} \subseteq 2^{[n]}$ such that for all $A, A' \in \mathcal{A}$, neither $A \subsetneq A'$, nor $A' \subsetneq A$ (such $\mathcal{A}$ is called an *anti-chain of the boolean lattice*). Then $\#\mathcal{A} \leq \binom{n}{\lfloor n/2 \rfloor}$.".

In the following, a *Sperner family* is a family $\mathcal{A} \subseteq 2^{[n]}$ such that for all $A, A' \in \mathcal{A}$, $A \not\subset A'$ and $A' \not\subset A$. We define $\mathcal{A}_k = \mathcal{A} \cap \binom{[n]}{k}$ and $f_k = \#\mathcal{A}_k$.

1. Prove Sperner theorem when $\mathcal{A} \subseteq \binom{[n]}{k}$ for some k.
2. (Lubell–Yamamoto–Meshalkin inequality) Double-count the pairs (X, σ) where $X \in \mathcal{A}$ and σ is a permutation of $[n]$ with $X = \{\sigma(1), \ldots, \sigma(\#X)\}$. Deduce that $\sum_{k=0}^{n} \frac{1}{\binom{n}{k}} f_k \leq 1$.
3. Using a clever bound on $\binom{n}{k}$, deduce Sperner theorem.
4. Consider that biggest ℓ such that $\mathcal{A}_\ell \neq \emptyset$, and replace it by its lower shadow. Show that $\#\Delta\mathcal{A}_\ell \geq \frac{\ell}{n-\ell+1} \#\mathcal{A}_\ell$.
5. Show that replacing $\mathcal{A}_\ell$ by $\Delta\mathcal{A}_\ell$, we still have a Sperner family.

Exercise 43 [Lovász formulation ⋆]
Let $\mathcal{A} \subseteq \binom{[n]}{k}$, and let $\mathcal{B}$ be the collection of the subsets of $\mathcal{A}$ of size $k-r$. Lovász claims that, for any $x > 0$, if $\#\mathcal{A} = \binom{x}{k}$, then $\#\mathcal{B} \geq \binom{x}{k-r}$.

Prove it using Kruskal–Katona theorem.

Recall that for any $x > 0$, we have: $\binom{x}{k} = \frac{x(x-1)\ldots(x-k+1)}{k!}$, even when x is not an integer.

Exercise 44 [r-neighborhood in graphs ⋆]
In a graph G, the *r-neighborhood* of a vertex $v \in V(G)$ is the "ball" of radius r around v, that is $N^r = \{u \in V(G) \; ; \; \mathrm{dist}(v, u) \leq r\}$. The r-neighborhood of a set S of vertices of G is naturally $N^r(S) = \{u \in V(G) \; ; \; \exists v \in S, \, \mathrm{dist}(v, u) \leq r\}$.

Let Q_n be the graph of the n-dimensional cube. Let $\mathcal{A} \subseteq V(Q_n)$ such that $\#\mathcal{A} \geq \sum_{i=0}^{k} \binom{n}{i}$. Show that $\#N^r(\mathcal{A}) \geq \sum_{i=0}^{k+r} \binom{n}{i}$.

Exercise 45 [Iterated Kruskal–Katona ◦]
For $\mathcal{A} \subseteq \binom{[n]}{k}$, we define the ℓ-shadow as: $\Delta_\ell \mathcal{A} = \{B \in \binom{[n]}{\ell} \; ; \; \exists A \in \mathcal{A}, \, B \subseteq A\}$.

Suppose $\#\mathcal{A} = m = \binom{a_k}{k} + \ldots \binom{a_j}{j}$ with $a_k > \cdots > a_j \geq j \geq 1$. What is the smallest size of the ℓ-shadow of $\mathcal{A}$ for $\ell \in [0, k]$?

Re-read Erdös–Ko–Rado theorem and comment.

Exercise 46 [Stronger LYM using Kruskal–Katona ⋆]
Fix $(a_0, \ldots, a_n)$, and construct $(w_0, \ldots, w_n)$ with $w_n = a_n$ and $w_k = KK(w_{k+1}, k+1) + a_k$, where $KK(x, \ell)$ denotes the Kruskal–Katona ℓ-reduction of x (i.e. if $x = \binom{b_\ell}{\ell} + \cdots + \binom{b_j}{j}$, then $KK(x, \ell) = \binom{b_\ell}{\ell-1} + \cdots + \binom{b_j}{j-1}$).

Recall that $\mathcal{A} \subseteq 2^{[n]}$ is an anti-chain if for all $A, A' \in \mathcal{A}$: $A \not\subset A'$ and $A' \not\subset A$.

Prove that there is an anti-chain $\mathcal{A} \subseteq 2^{[n]}$ with exactly $\#(\mathcal{A} \cap \binom{[n]}{k}) = a_k$ for all $0 \leq k \leq n$ if and only if $w_1 \leq n$ and $w_0 \leq 1$.

Exercise 47 [Kneser graphs $\star$]

The *Kneser graph* $KG(n,k)$ is the graph with vertex set $\binom{[n]}{k}$, and edges between A and B if $A \cap B = \emptyset$.

1. Draw $KG(5,2)$ and try to remember which name people usually give it.
2. For $2k > n$, what is $KG(n,k)$?
3. Show that the chromatic number of the Kneser graph is $\chi(KG(n,k)) = n - 2k + 2$ for $2k \leq n$. [This is particularly hard at this point: I would need to make more sub-questions...]

For a graph G, denoting $\mathcal{I}_G$ the collection of its independent sets (i.e. sub-set of its vertices pair-wise not adjacent), the *fractional chromatic number* $\chi_f(G)$ is the minimum $r \in \mathbb{R}$ such that for each $I \in \mathcal{I}_G$, there exists $x_I \geq 0$ satisfying $\sum_{I \in \mathcal{I}_G} x_I = r$ and for every vertex $v \in V(G)$ we have $\sum_{I \ni v} x_f \geq 1$.

4. Using $n = \#V(G)$ and $\alpha(G) = \max_{I \in \mathcal{I}_G} \#I$, show: $\frac{n}{\alpha(G)} \leq \chi_f(G) \leq \chi(G)$.
5. For $n \geq 2k$, show that $\chi_f(KG(n,k)) = \frac{n}{k}$.

Exercise 48 [Shifting a simplicial complex $\star$]

Let Δ be a simplicial complex. Recall that a simplicial complex is shifted if for every face F with $i \in F$ and $j < i$ it follows that $(F \backslash \{i\}) \cup \{j\} \in \Delta$.

(i) Prove that a pure shifted simplicial complex is shellable. *Hint*: Consider the lexicographic order on the facets.

Exercise 49 [Equality case of Erdös–Ko–Rado $\circ$]

Study the equality case of Erdös–Ko–Rado's theorem.

Exercise 50 [Using Erdös–Ko–Rado $\star$]

Let $\mathcal{A} \subseteq \binom{[n]}{k}$ with $k \geq \frac{n}{2}$ such that for all $A, A' \in \mathcal{A}$, we have $A \cup A' \neq [n]$. Show that: $\#\mathcal{A} \leq \binom{n-1}{k}$.

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Simplicial Complexes – Summer Semester 2025

Repetitorium 3

Exercise 51 [Cyclic polytope and realization of a simplicial complex ∘]
Show that a simplicial complex of dimension d is realizable in $\mathbb{R}^{2d+1}$.

Find a simplicial complex of dimension d which is realizable in $\mathbb{R}^{2d}$.

Find a simplicial complex of dimension d which is **not** realizable in $\mathbb{R}^{2d}$.

Exercise 52 [Other definition of faces of a polytope ∘]
For a polytope $\mathsf{P} \subseteq \mathbb{R}^d$ and a vector $\boldsymbol{c} \in \mathbb{R}^d$, we define $\mathsf{P}^{\boldsymbol{c}} = \{\boldsymbol{x} \in \mathsf{P} \; ; \; \langle \boldsymbol{x}, \boldsymbol{c} \rangle = \max_{\boldsymbol{y} \in \mathsf{P}} \langle \boldsymbol{y}, \boldsymbol{c} \rangle\}$.
Show that $\boldsymbol{F}$ is a face of P if and only if there exists $\boldsymbol{c} \in \mathbb{R}^d$ such that $\boldsymbol{F} = \mathsf{P}^{\boldsymbol{c}}$.

Exercise 53 [Simple and simplicial ∘]
Which polytopes are simple and simplicial.

Exercise 54 [Dual polytope ⋆]
What is the polar of the regular n-gon? What is the polar of the standard d-simplex? What is the polar of the standard d-cube? Of the standard d-cross-polytope?

For a face F of P with $\mathbf{0} \in \text{int}(\mathsf{P})$, consider $\mathsf{F}^{\diamond} := \left\{ \boldsymbol{y} \in \mathbb{R}^d \; ; \; \begin{array}{l} \forall \boldsymbol{x} \in \mathsf{P}, \; \langle \boldsymbol{x}, \boldsymbol{y} \rangle \leq 1 \\ \forall \boldsymbol{x} \in \mathsf{F}, \; \langle \boldsymbol{x}, \boldsymbol{y} \rangle = 1 \end{array} \right\}$.
Show that $\mathsf{F}^{\diamond}$ is a face of P^*.

Show that $\left(\mathsf{P}^*\right)^* = \mathsf{P}$, and $\left(\mathsf{F}^{\diamond}\right)^{\diamond} = \mathsf{F}$.

Show that the lattices of faces of P and of P^* are dual (i.e. opposite) to each other.

Exercise 55 [Simplicial/simple polytopes ⋆]
Let P be a polytope of dimension d. Show that the following are equivalent:

1. Every facet of P has d vertices.
2. Every proper face of P is a simplex.
3. Every k-face of P has $k+1$ vertices for $k \leq d-1$.

Show that dual version, i.e. the following are equivalent:

a. Every vertex of P lies in d facets.
b. Every vertex of P in d edges.
c. Every k-face of P lies in $d-k$ facets for $k \geq 0$.

Let v be a simplex vertex of P, and let P/v be a *vertex figure* of v in P, that is a polytope obtained by intersecting P with an hyperplane which separate v from all other vertices of P. Show that P/v is a simplex.

Deduce that for any subset of edges adjacent to v, there exists a face $\boldsymbol{F}$ of P which contains exactly theses edges (and no other edge adjacent to v).

Exercise 56 [Random polytope ∘]
Pick n points $X_1, \dots, X_n$ at random in $\mathbb{R}^d$ (say, for the uniform distribution in the cube). Is $\operatorname{conv}(X_1, \dots, X_n)$ simple or simplicial?

Pick m vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$ at random in $\mathbb{R}^d$ (say, for the uniform distribution on the sphere). Is $\{\boldsymbol{x} \in \mathbb{R}^d \ ; \ \forall i,\ \langle \boldsymbol{x}, \boldsymbol{a}_i \rangle \leq 1\}$ simple or simplicial?

Exercise 57 [Set of f-vectors ⋆]
(Pretend to) write the Dehn–Sommerville equation for the f-vector. What type of equation is it (quadratic, polynomial, exponential, etc)?

Consider the smallest vector space containing all f-vectors of polytopes: what is its dimension, at most?

Fix d. Show that the f-vectors of $\mathsf{Cyc}(n,d)^*$ are linearly independent, for $d+1 \leq n \leq d + \lfloor \frac{d}{2} \rfloor + 1$.

Conclude that the Dehn–Sommerville equations are the only (linear) equations between f-vectors of simple polytopes.

Exercise 58 [Euler relation for simple polytopes ∘]
Using Dehn–Sommerville equations and $F(X-1) = H(X)$, deduce the Euler characteristic (for simple polytopes):

$$f_0 - f_1 + \cdots + (-1)^{d-1} f_{d-1} = 2$$

This relation actually holds for any sphere (it is provable using reduced homology + topology).

Exercise 59 [Euler relation is the only relation ⋆]
Let $\mathsf{Pyr}(\mathsf{P}) = \operatorname{conv}\{(\boldsymbol{v}, 0) \ ; \ \boldsymbol{v} \in \mathsf{P}\} \cup \{(\boldsymbol{0}, 1)\}$. Compute the f-vector of $\mathsf{Pyr}(\mathsf{P})$ from the f-vector of P.

Define $\mathsf{Pyr}^{i+1}(\mathsf{P}) = \mathsf{Pyr}^i(\mathsf{Pyr}(\mathsf{P}))$ and $\mathsf{Pyr}^0(\mathsf{P}) = \mathsf{P}$. Show that the f-vectors of $\mathsf{Pyr}^i([0,1]^{d-i})$ are linearly independent. Deduce that the Euler relation is the only linear equation on f-vectors of polytopes of dimension d.

Exercise 60 [Trivial inequality ∘]
For any polytope on n vertices, show that $f_i \leq \binom{n}{i+1}$.

Exercise 61 [Lower bound theorem (no proof) ⋆]
For a polytope P and a facet F, a *vertex stacking* of F in P is a polytope $\operatorname{conv}(\mathsf{P} \cup \{\boldsymbol{x}\})$ where $\boldsymbol{x}$ (strictly) satisfies all the inequalities that define P but does not satisfy the inequality defining F. A *n-stacked polytope* $\Delta_{d-1}^{\uparrow n-d}$ is a polytope obtained by stacking $n-d$ times on a d-simplex.

Compute the f-vector of a stacking in term of the vector of P.

Prove that a n-stacked polytope is simplicial, and compute its f-vector (and show this independently from the chosen sequence of stackings). In particular, show it has n vertices.

Admits and contemplate the Lower Bound Theorem:

For all simplicial polytopes P on n vertices:

$$f_i(\mathsf{P}) \geq f_i(\Delta_{d-1}^{\uparrow n-d})$$

Exercise 62 [f-vector of 3-polytopes]
Let P be a 3-polytopes, show that its f-vector can be recovered from f_0 and f_2 only.

In the plane, draw two orthogonal axes, the horizontal being labeled f_0 and the vertical f_2. Each axis goes from 0 to 10. Place on this graphic the f-vector of a 3-simplex, a 3-cube, a 3-cross-polytope (octahedron). Place the f-vectors of pyramids over n-gons, of prism over n-gons.

At least, how many edges can be adjacent to a vertex of a 3-polytopes? Deduce that $f_0 \leq 2f_2 - 4$, and that the equality is satisfied by simple 3-polytopes. Draw the line $f_0 = 2f_2 - 4$ on your graphic.

Dually, at least of how many vertices shall a facet of a 3-polytope contain? Deduce that $f_2 \leq 2f_0 - 4$, and that the equality is satisfied by simplicial 3-polytopes. Draw the line $f_2 = 2f_0 - 4$ on your graphic.

You now know that the f-vector of a 3-polytope correspond to an integer point of the cone C you have drawn (defined by $f_0 \leq 2f_2 - 4$ and $f_2 \leq 2f_0 - 4$). It remains to prove the reciprocal: for each integer point of the cone C, there exists a polytope whose f-vector correspond to this point.

Interpret $\mathsf{P} \mapsto \mathsf{P}^*$ on your graphic.

Stack a vertex on the facet of a simplicial 3-polytope, what happens to its f-vector?

For a polytope P and a vertex $\boldsymbol{v} \in V(\mathsf{P})$, a *vertex truncation* of $\boldsymbol{v}$ in P is a polytope defined by intersecting P with $H^+_{\boldsymbol{a},b-\varepsilon}$ for $H_{\boldsymbol{a},b}$ a supporting hyperplane of $\boldsymbol{v}$. Equivalently, a vertex truncation is $\mathrm{conv}\big(\mathsf{P}\backslash\boldsymbol{v}\big)\cup\{\boldsymbol{w} \in v(\mathsf{P}) \ ; \ \boldsymbol{w} \neq \boldsymbol{v}\}$.

By starting with a 3-cross-polytope, then cleverly doing a series of vertex stacking, then doing a clever vertex truncation, construct a 3-polytope for each integer point in the cone C. This proves the following the Theorem:

“ A vector $(1, f_0, f_1, f_2, 1)$ is the f-vector of a 3-polytope, if and only if:

$$f_0 - f_1 + f_2 = 2 \quad , \quad f_0 \leq 2f_2 - 4 \quad \text{and} \quad f_2 \leq 2f_0 - 4$$

”

Find two 3-polytopes with different face lattices but the same f-vector.

Where is the set of f-vector corresponding to 3-polytopes with k edges for a given k.

Show that $F_1 := \{f_1 \ ; \ (1, f_0, f_1, f_2, 1) \text{ the } f\text{-vector of a 3-polytope}\}$ is not (the integral points of a) convex set, by looking at 3-polytopes with 7 edges.

Compute F_1 explicitly.

Exercise 63 [Which ideals are graded? ◦]
Let $I \subseteq \mathbb{F}[x_1, \dots, x_n]$ be an ideal. Show that I is graded if and only if it is generated by homogeneous elements.

Exercise 64 [Have you read the lecture notes? ∘]
Prove Theorem 2.6 from the lecture.

Exercise 65 [Chains of modules ⋆]
Let R be a finitely generated graded $\mathbb{F}$-algebra, $M_0, M_1, \ldots, M_n$ finitely generated graded R-modules, and

$$0 \to M_n \xrightarrow{\phi_n} M_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_2} M_1 \xrightarrow{\phi_1} M_0 \to 0$$

an exact sequence, i.e., the ϕ_i are homogeneous R-module homomorphisms with $\operatorname{im}(\phi_i) = \ker(\phi_{i-1})$ for all i. In particular, ϕ_n is injective and ϕ_1 is surjective. Show that $\sum_{i=0}^{n}(-1)^i H(M_i, j) = 0$ holds for all $j \in \mathbb{Z}$.

Exercise 66 [Noetherian rings ⋆]
Let R be a commutative ring. An R-module M is called *Noetherian* if every submodule of M is finitely generated. Show:

(a) Let M, N be two R-modules and let $\phi\colon M \to N$ be an R-module homomorphism. Then $\ker(\phi)$, $\phi^{-1}(N)$ and $\phi(M)$ are R-modules again. If $\ker(\phi)$ and $\phi(M)$ are finitely generated, then M is finitely generated.

(b) Let R be a Noetherian ring. Then a finitely generated free R-module is Noetherian.

(c) Let R be Noetherian and M an R-module. Then M is Noetherian if and only if M is finitely generated.

Exercise 67 [Graded rings ∘]
Let R be a finitely generated standard graded $\mathbb{F}$-algebra. Show:

(a) $\dim R = 0$ if and only if $R_i = 0$ for $i \gg 0$;

(b) $\dim R = 1$ if and only if there exists a $c \in \mathbb{N}, c > 0$ with $\dim_{\mathbb{F}} R_i = c$ for $i \gg 0$.

Exercise 68 [Hilbert polynomial ⋆]
Let R be a finitely generated standard graded K-algebra, and $0 \neq M$ a finitely generated graded R-module. Show that there exists a polynomial $P_M \in \mathbb{Q}[t]$ such that

$$P_M(i) = H(M, i) \text{ for } i \gg 0.$$

If $\dim M = 0$, then $P_M(t) = 0$. If $\operatorname{Hilb}(M; t) = \frac{Q(t)}{(1-t)^{\dim M}}$ with $Q(1) \neq 0$ and $\dim M > 0$, then you can conclude that

$$P_M(t) = \frac{Q(1)}{(\dim M - 1)!} T^{\dim M - 1} + (\text{ lower degree terms in } t).$$

P_M is called the *Hilbert polynomial* of M.

Exercise 69 [Noether normalization ⋆]
Let $S = \mathbb{F}[x, y, z]$, $I = (xy, xz^2, z^3) \subset S$ and $R = S/I$. Determine a Noether normalization of R.

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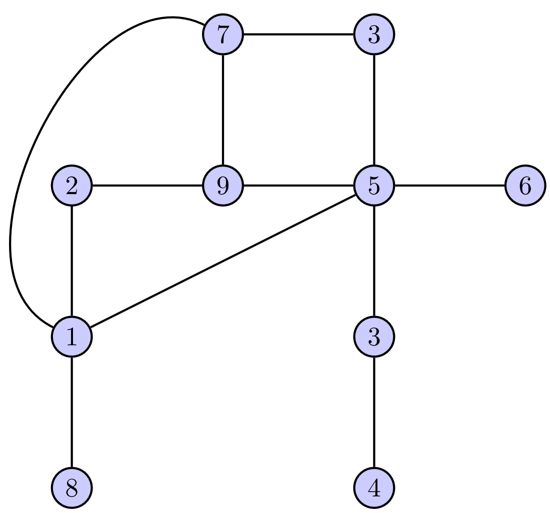
Simplicial Complexes – Summer Semester 2025

Repetitorium 4

Exercise 70 [Clique complex ⋆]

- Given the simplicial complex on the vertex set $[6]$ with facets $\{1, 2, 3, 4\}$, $\{1, 2, 5\}$, and $\{5, 6\}$, show that Δ is a clique complex and determine the corresponding graph.
- Given the simplicial complex on the vertex set $[6]$ with facets $\{1, 2, 3, 4\}$, $\{1, 5\}$, $\{2, 5\}$, and $\{5, 6\}$, justify why this is not a clique complex.

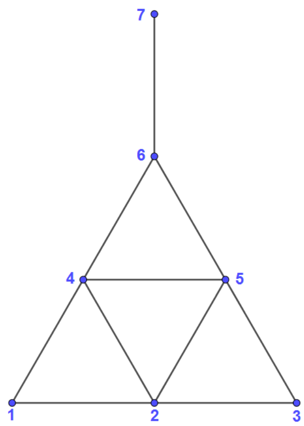
Exercise 71 [Cohen-Macaulay graph ∘]
Check whether the following graph is Cohen–Macaulay.



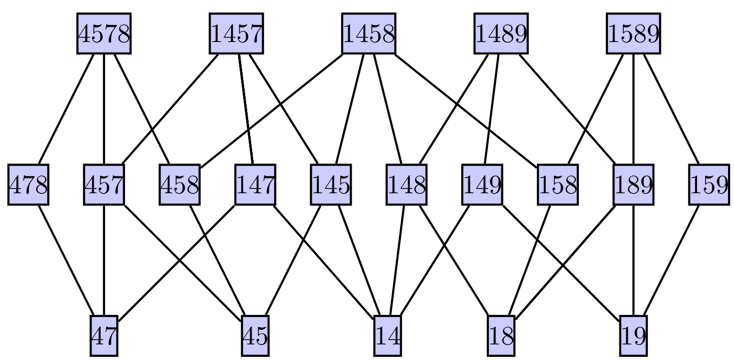
Exercise 72 [Dirac's theorem ∘]
Show that the following graph G satisfies all conditions of Dirac's theorem for chordal graphs:

Exercise 73 [Cohen-Macaulay-ness and poset ⋆]
Given the poset Φ (see diagram below):

- Show that Φ is a relatively Cohen–Macaulay complex.



- Determine the minimal representation $\Phi = (\Delta, \Gamma)$ with simplicial complexes $\Delta \subseteq \Gamma$.
- Is Γ an induced subcomplex of Δ? Justify your answer.
- Show that Φ is not partitionable.



Discrete Mathematics – Sheet 1

Exercise 1 [Counting simplicial complexes]

A pure d-dimensional simplicial complex on n vertices can be seen as a collection of facets, which is a (non-empty) collection of subsets of $[n]$ of cardinal $d+1$. There are $\binom{n}{d+1}$ subsets of $[n]$ of cardinal $d+1$. Hence, there are $2^{\binom{n}{d+1}}$ collections of such subsets; 1 of these collections is empty and shall not be counted.

Exercise 3 [Join]

We denote by $X \sqcup Y$ the union $X \cup Y$ when $X \cap Y = \emptyset$.

It is a simplicial complex If $E \in \Delta \star \Gamma$, then there exists $F \in \Delta$ and $G \in \Gamma$ such that $E = F \sqcup G$.

Suppose $E' \subseteq E$, then there exists $F' \subseteq F$ and $G' \subseteq G$ such that $E' = F' \sqcup G'$ (because E is a disjoint union). As Δ and Γ are simplicial complexes, we have $F' \in \Delta$ and $G' \in \Gamma$. Thus, be definition, $F' \sqcup G' \in \Delta \star \Gamma$.

Consequently, $\Delta \star \Gamma$ is a simplicial complex. Moreover, $\dim(\Delta \star \Gamma) = \dim \Delta + \dim \Gamma + 1$ (we will denote $d = d_\Delta + d_\Gamma + 1$).

f-vector and f-polynomial We obviously have $\left|F \sqcup G\right| = |F| + |G|$. The number of $(k-1)$-faces of $\Delta \star \Gamma$ is its number of faces of cardinality k:

$$f_{k-1}^{\Delta \star \Gamma} = \sum_{i+j=k} f_{i-1}^{\Delta} \cdot f_{j-1}^{\Gamma} = \sum_{i=0}^{k} f_{i-1}^{\Delta} \cdot f_{k-i-1}^{\Gamma}$$

Consequently, the f-polynomial of $\Delta \star \Gamma$ is:

$$\begin{aligned}
F^{\Delta \star \Gamma}(X) &= \textstyle\sum_{k=0}^{d+1} f_{k-1}^{\Delta \star \Gamma} X^{d-(k-1)} \\
&= \textstyle\sum_{k=0}^{d+1} \left(\sum_{i+j=k} f_{i-1}^{\Delta} \cdot f_{j-1}^{\Gamma} \right) X^{d-(k-1)} \\
&= \left(\textstyle\sum_{i=0}^{d_\Delta+1} f_{i-1}^{\Delta} X^{d_\Delta-(i-1)} \right) \cdot \left(\textstyle\sum_{j=0}^{d_\Gamma+1} f_{j-1}^{\Gamma} X^{d_\Gamma-(j-1)} \right) \\
&= F^{\Delta}(X) \cdot F^{\Gamma}(X)
\end{aligned}$$

h-polynomial and h-vector As we know the f-polynomial, it is easy to deduce the h-polynomial:

$$H^{\Delta \star \Gamma}(X) = F^{\Delta \star \Gamma}(X-1) = H^{\Delta}(X) \cdot H^{\Gamma}(X)$$

Thus: $h_{d-(k-1)}^{\Delta \star \Gamma} = \sum_{i+j=k} h_{d_\Delta-(i-1)}^{\Delta} \cdot h_{d_\Gamma-(j-1)}^{\Gamma}$.

Exercise 13 [Truncated boundary of a simplex]

Note that $F \subseteq F_i$ if and only if $i \notin F$. Conversely, $F \not\subseteq F_i$ if and only if $i \in F$.

Stanley-Reisner ideal Hence, if for all $i \in [m]$, $F \not\subseteq F_i$, then then for all i, $i \in F$. Equivalently, F is a non-face of Δ if and only if $[m] \subseteq F$.

So Δ has a unique minimal non-face, namely $[m]$, and its Stanley-Reisner ring is $\mathbb{F}[\Delta] = \mathbb{F}[X_1, \ldots, X_n]/(X_1 \ldots X_m)$.

***h*-vector, method 1** We start with the f-vector: $f_{k-1} = \binom{n}{k} - \binom{n-m}{k-m}$ because choosing a $(k-1)$-face amounts to choosing among the subsets of $[n]$ of size k, but not among the subsets of $[n]$ of size k which contains $[m]$ (these subsets are in bijection with the subsets of $[m+1, n]$ of size $k-m$).

Thus, we can compute the f-polynomial (note that $\dim \Delta = |F_1| - 1 = n-2$):

$$\begin{aligned} F(X) &= \textstyle\sum_{k=0}^{n-1} f_{k-1}\, X^{n-1-k} \\ &= \textstyle\sum_{k=0}^{n-1} \binom{n}{k} X^{n-1-k} - \sum_{k=m}^{n-1} \binom{n-m}{k-m} X^{n-1-k} \\ &= \textstyle\sum_{k=1}^{n} \binom{n}{k} X^{k} - X^{n-m-1} \sum_{k=0}^{n-m-1} \binom{n-m}{k} \left(\frac{1}{X}\right)^{k} \\ &= \tfrac{1}{X}\Big((X+1)^n - 1\Big) - X^{n-m-1}\left(\left(\tfrac{1}{X}+1\right)^{n-m-1} - \left(\tfrac{1}{X}\right)^{n-m}\right) \\ &= \tfrac{1}{X}\Big((X+1)^n - (X+1)^{n-m}\Big) = \tfrac{(X+1)^m - 1}{X}(X+1)^{n-m} \end{aligned}$$

Thus: $H(X) = F(X-1) = \frac{X^{n-m}-1}{X-1} X^{n-m}$. Note that $\frac{X^{n-m}-1}{X-1} = \sum_{k=0}^{n-m-1} X^k$, so:

$$h_{k-1} = \begin{cases} 1 & \text{for } 1 \le k \le m \\ 0 & \text{else} \end{cases}$$

***h*-vector, method 2** [Proposed by Anna Birkemeyer] Remark that Δ is shellable: $F_1, \ldots, F_m$ is already a shelling order.

Let $G_i = [i-1]$. Then we have the partition:

$$\Delta = \bigsqcup_i [G_i,\, F_i]$$

Indeed, if $F \in \Delta$, then let $j = \min(i \in [n]\ ;\ i \notin F)$. Then $G_j \subseteq F \subseteq F_j$, but if $i < j$, then $F \not\subseteq F_i$, and if $i > j$, then $G_j \not\subseteq F$: the above union is indeed a partition of Δ (each face is contained in a unique $[G_i,\, F_i]$).

We can use the theorem 1.26 (on h-vector of partiotionable simplicial complexes) from the lecture: $h_i = \big|\{j\ ;\ |G_j| = i\}\big| = \begin{cases} 1 & \text{if } 0 \le i \le m-1 \\ 0 & \text{else} \end{cases}$.

Exercise 32/33 [Triangulations and homology]

You should obtain (all other $\tilde{H}_i$ are $\mathbf{0}$):

	char($\mathbb{F}$) $\neq 2$		char($\mathbb{F}$) $= 2$	
	$\tilde{H}_1$	$\tilde{H}_2$	$\tilde{H}_1$	$\tilde{H}_2$
Torus	$\mathbb{F}^2$	$\mathbb{F}$	idem	idem
Klein bottle	$\mathbb{F}$	$\mathbf{0}$	$\mathbb{F}^2$	$\mathbb{F}$
Projective plane	$\mathbb{F}$	$\mathbf{0}$	$\mathbb{F}^2$	$\mathbb{F}$
Cylinder	$\mathbb{F}$	$\mathbf{0}$	idem	idem
Möbius strip	$\mathbb{F}$	$\mathbf{0}$	idem	idem
Sphere $\mathbb{S}^2$	$\mathbf{0}$	$\mathbb{F}$	idem	idem
Ball	$\mathbf{0}$	$\mathbf{0}$	idem	idem

N.B.: Be very careful, you may find online some different results for the Klein bottle, the projective plane and the Möbius strip. That's expected: it comes from non-orientability problems...