Mathematics

Combinatorics of polytopes & Combinatorics with polytopes

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Contents

CHAPTER

1

INTRODUCTORY NOTIONS, f-VECTORS

Content of the chapter

Attention 1.0.1 *Recall that:*

$\mathbb{R}^d$ is a vector space of finite dimension d. Usually, the vectors are denoted in bold $\boldsymbol{v} \in \mathbb{R}^d$. The canonical basis of $\mathbb{R}^d$ is denoted $\boldsymbol{e}_1, \dots, \boldsymbol{e}_d$, and for $X \subseteq [n]$ we write $\boldsymbol{e}_X := \sum_{i \in X} \boldsymbol{e}_i$. The space $\mathbb{R}^d$ is endowed with a scalar product denoted $\langle \boldsymbol{u}, \boldsymbol{v} \rangle := \sum_{i=1}^d u_i v_i \in \mathbb{R}$ for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d$. In reality, almost all we do (especially the computations) will be done in $\mathbb{Q}^d$ or even $\mathbb{Z}^d$, but we still use $\mathbb{R}^d$ in definitions and theorems.

Notation 1.0.2 *We use the word "vector" when we think about a linear problem, "point" when we think about affine geometry, "direction" when we think about a vector in the dual.*

We denote $[n] = \{1, 2, \dots, n\}$.

Attention 1.0.3 *The aim of the exercise is to understand the notions at stake. None of them will be graded (except if we need to for administrative reasons). If you manage to do an exercise without making a nice drawing, then you should re-do it! More generally, this course contains very few drawings, the aim being that you (i.e. the reader, the learner) make your own drawings.*

Most of the proofs of the theorem claimed will be done in exercises.

Section 1.1

Basic notions, convexity, Minkowski-Weyl

Definition 1.1.1
A set $X \subseteq \mathbb{R}^d$ is convex if $\forall \boldsymbol{x}, \boldsymbol{x}' \in X$ and $\lambda \in [0,1]$, then $\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{x}' \in X$. The convex hull of a subset $X \subseteq \mathbb{R}^d$ is the smallest convex set that contains X, that is to say: $\operatorname{conv} X = \bigcap_{X \subseteq Y,\ Y \text{ convex}} Y = \{\sum_i \lambda_i \boldsymbol{x}_i \ ;\ \lambda_i \geq 0,\ \sum_i \lambda_i = 1,\ \boldsymbol{x}_i \in X\}$.

⋆ **Exercise 1.1.2**
Make a drawing of a convex an a non-convex set in $\mathbb{R}^2$.

Definition 1.1.3
A polytope is defined equivalently as:

- *The convex hull of* **finitely** *many points $\mathsf{P} := \operatorname{conv}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ for $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n \in \mathbb{R}^d$.*
- *The* **bounded** *intersection of* **finitely** *many half-spaces $\mathsf{P} := \bigcap_{i=1}^m \{\boldsymbol{x} \in \mathbb{R}^d \ ;\ \langle \boldsymbol{x}, \boldsymbol{a}_i \rangle \leq b_i\}$ for $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m \in \mathbb{R}^d$ and $b_1, \dots, b_m \in \mathbb{R}$. We usually denote the vector $\boldsymbol{b} = (b_1, \dots, b_m)$, and the matrix A whose rows are $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$, such that $\mathsf{P} := \{\boldsymbol{x} \in \mathbb{R}^d \ ;\ A\boldsymbol{x} \leq \boldsymbol{b}\}$ where the comparison is coordinate-wise.*

The first is called the vertex description and the second the inequality (or facet) description of P.

⋆ **Exercise 1.1.4**
Is the euclidean ball a polytope? Why?
Is a cone a polytope? Why?

Theorem 1.1.5 — [Minkowski-Weyl]
Both definitions of polytopes are equivalent, that is to say: There exists $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ such that $\mathsf{P} = \operatorname{conv}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ if and only if there exists $A \in \mathbb{R}^{m \times d}$, $\boldsymbol{b} \in \mathbb{R}^d$ such that $\mathsf{P} := \{\boldsymbol{x} \in \mathbb{R}^d \ ;\ A\boldsymbol{x} \leq \boldsymbol{b}\}$.

Definition 1.1.6 — [Affine independence, Dimension]
Points $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ are affinely independent when $\begin{pmatrix} 1 \\ \boldsymbol{v}_1 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ \boldsymbol{v}_n \end{pmatrix}$ are linearly independent, that is to say if $\sum_i \lambda_i = 0$ and $\sum_i \lambda_i \boldsymbol{v}_i = \boldsymbol{0}$, then $\lambda_i = 0$ for all i.

The (affine) dimension of a subset $X \subseteq \mathbb{R}^d$ is the cardinal of a maximal affinely independent subset of X. A polytope of dimension d is usually called a d-polytope.

The affine span of $X \subseteq \mathbb{R}^d$ is $\operatorname{aff}(X) := \{\sum_i \lambda_i \boldsymbol{x}_i \ ;\ \sum_i \lambda_i = 1,\ \boldsymbol{x}_i \in X\}$.

Definition 1.1.7 — [Standard simplex, Standard cube, Standard cross-polytope, Regular polygon]
The standard $(d-1)$-simplex is $\Delta_{d-1} := \operatorname{conv}\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_d\} \subset \mathbb{R}^d$.
The standard d-cube is $\square_d := \operatorname{conv}\{\boldsymbol{e}_X \ ;\ X \subseteq [d]\}$.
The standard d-cross-polytope is $\diamondsuit_d := \operatorname{conv}\{\boldsymbol{e}_1, -\boldsymbol{e}_1, \dots, \boldsymbol{e}_d, -\boldsymbol{e}_d\} \subset \mathbb{R}^d$.
The (d,k)-hypersimplex is $\Delta(d,k) := \operatorname{conv}\{\boldsymbol{e}_X \ ;\ X \subseteq [d],\ |X| = k\} \subset \mathbb{R}^d$.
The regular n-gon is $\operatorname{conv}\big\{(\cos\frac{2k\pi}{n}, \sin\frac{2k\pi}{n}) \ ;\ k \in [n]\big\} \subset \mathbb{R}^2$

⋆ **Exercise 1.1.8**
What are the dimension of the standard $(d-1)$-simplex, d-cube and d-cross-polytope, the (d,k)-hypersimplex, the regular n-gon?
Give a inequality description of the $(d-1)$-simplex, d-cube and d-cross-polytope, the (d,k)-hypersimplex, the regular n-gon?

Section 1.2

Faces, face lattice, f-vector

Definition 1.2.1 — [Faces]
For a polytope $\mathsf{P} \subset \mathbb{R}^d$ and a direction $\boldsymbol{c} \in R$, the face of P in direction $\boldsymbol{c}$ is:

$$\mathsf{P}^{\boldsymbol{c}} := \{\boldsymbol{x} \in \mathsf{P} \ ; \ \langle \boldsymbol{x}, \boldsymbol{c}\rangle = \max_{\boldsymbol{y}\in\mathsf{P}} \langle \boldsymbol{y}, \boldsymbol{c}\rangle\}$$

The set of all faces of P is $\mathcal{F}(\mathsf{P}) = \{\varnothing\} \cup \{\mathsf{P}^{\boldsymbol{c}} \ ; \ \boldsymbol{c} \in \mathbb{R}^d\}$.

Remark 1.2.2 Yes, by convention, $\varnothing$ is considered a face of P.

Definition 1.2.3 — [Valid hyperplane]
An hyperplane $H_{\boldsymbol{a},b} = \{\boldsymbol{x} \in \mathbb{R}^d \ ; \ \langle \boldsymbol{x}, \boldsymbol{a}\rangle = b\}$ naturally define two half-spaces $H^+_{\boldsymbol{a},b} = \{\boldsymbol{x} \in \mathbb{R}^d \ ; \ \langle \boldsymbol{x}, \boldsymbol{a}\rangle \geq b\}$ and $H^-_{\boldsymbol{a},b} = \{\boldsymbol{x} \in \mathbb{R}^d \ ; \ \langle \boldsymbol{x}, \boldsymbol{a}\rangle \leq b\}$. Such an hyperplane is called a valid hyperplane for a polytope P when either $\mathsf{P} \subseteq H^+_{\boldsymbol{a},b}$ or $\mathsf{P} \subseteq H^-_{\boldsymbol{a},b}$.

⋆ **Exercise 1.2.4**
How many faces does a d-simplex, a d-cube, a d-cross-polytope, the (d, k)-hypersimplex, a n-gon has?

⋆ **Exercise 1.2.5**
Show that for all proper faces F of P, there exists a valid hyperplane $H_{\boldsymbol{a},b}$ with $\mathsf{F} = \mathsf{P} \cap H_{\boldsymbol{a},b}$. Such an hyperplane is called a *supporting hyperplane* for F.

Theorem 1.2.6 *A face of a polytope is a polytope.*

⋆ **Exercise 1.2.7**
Prove this theorem.

⋆ **Exercise 1.2.8**
Let F be a non-empty face of P. Show that $\mathsf{F} = \mathsf{P} \cap \mathrm{aff}(\mathsf{F})$, where $\mathrm{aff}(\mathsf{F})$ is the affine span of F.

Definition 1.2.9 — [Vertices, Edges, Facets, Ridges]
The dimension of a face $\mathsf{F} \subseteq \mathsf{P}$ is the dimension of F as a polytope.
By convention, $\dim \varnothing = -1$, it is the only face of dimension -1.
The faces of dimension 0 are called the vertices of P.
The faces of dimension 1 are called the edges of P.
The faces of dimension $\dim \mathsf{P} - 1$ (a.k.a. of co-dimension 1) are called the facets of P.
The faces of dimension $\dim \mathsf{P} - 2$ (a.k.a. of co-dimension 2) are called the ridges of P.

⋆ **Exercise 1.2.10**
How many faces of dimension k does a d-simplex, a d-cube, a d-cross-polytope, the (d, k)-hypersimplex, a n-gon has ?

Definition 1.2.11 — [Face lattice]
The face lattice of P is the poset (partially ordered set) $\mathscr{L}_{\mathsf{P}} := \big(\mathcal{F}(\mathsf{P}), \subseteq\big)$.

Notation 1.2.12 *A polytope is "a" simplex when it has the same face lattice as the one of the standard simplex, and likewise for "a" cube, "a" cross-polytope, etc. . . This notion will be further explored in Section 2.2 on equivalences between polytopes.*

⋆ **Exercise 1.2.13**
Draw the face lattice of the 3-simplex, a 3-cube, a 3-cross-polytope, the $(4, 2)$-hypersimplex, a n-gon.

⋆ **Exercise 1.2.14**
What is the difference between a 2-cube and a 2-cross-polytope? Between (n, k)-hypersimplex and $(n, n - k)$-hypersimplex?
Draw a very weird 2-cube, a weird 3-cube.
Prove that being a d-simplex is equivalent to being the convex hull of $d + 1$ affinely independent points.

Theorem 1.2.15 *The face lattice $\mathscr{L}(\mathsf{P})$ of a d-dimensional polytope P:*

- *has a unique minimal element $\varnothing$ and a unique maximal element P.*
- *is a lattice.*
- *is graded by the dimension, and has total rank $d + 1$.*
- *is atomic and co-atomic.*

⋆ **Exercise 1.2.16**
If you do not understand one of these words, ask about it!

⋆ **Exercise 1.2.17**
Prove that $\mathscr{L}(\mathsf{P})$ has a unique minimal element, and a unique maximal element.

⋆ **Exercise 1.2.18**
Prove that $\mathsf{F} \wedge \mathsf{G} = \mathsf{F} \cap \mathsf{G}$.

⋆ **Exercise 1.2.19**
Let $\boldsymbol{v}$ be a vertex of P, and $H_{\boldsymbol{a},b}$ a supporting hyperplane for the face $\{\boldsymbol{v}\}$ of P, *i.e.* $\mathsf{P} \subseteq H^-_{\boldsymbol{a},b}$ and $\mathsf{P} \cap H_{\boldsymbol{a},b} = \{\boldsymbol{v}\}$. A *vertex figure* $\mathsf{P}/\boldsymbol{v}$ of $\boldsymbol{v}$ in P is defined as $\mathsf{P} \cap H_{\boldsymbol{a},b-\varepsilon}$ for $\varepsilon > 0$ sufficiently small (precisely, let $b' = \max\{\langle \boldsymbol{u}, \boldsymbol{a}\rangle \ ; \ \boldsymbol{u} \text{ vertex of } \mathsf{P},\ \boldsymbol{u} \neq \boldsymbol{v}\}$, and pick ε such that $b' < b - \varepsilon < b$).

Show that the face lattice of a vertex figure is isomorphic to the sub-lattice $[\boldsymbol{v}, \mathsf{P}]$ of $\mathscr{L}_{\mathsf{P}}$.
Deduce that all interval $[\mathsf{F}, \mathsf{G}]$ of $\mathscr{L}_{\mathsf{P}}$ is isomorphic to the face lattice of some polytope.
Deduce that all intervals of length 2 in $\mathscr{L}_{\mathsf{P}}$ has 4 elements, and draw the shape of such interval.
Deduce that $\mathscr{L}_{\mathsf{P}}$ is graded by the dimension, and has total rank $\dim \mathsf{P} + 1$.

Definition 1.2.20 — [Dual polytope]
For a polytope P with vertex set $V(\mathsf{P}) = \mathrm{conv}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ such that $\boldsymbol{0} \in \mathrm{int}(\mathsf{P})$, its dual polytope (or polar polytope) is:

$$\mathsf{P}^\circ := \{\boldsymbol{y} \in \mathbb{R}^d \ ; \ \forall \boldsymbol{x} \in \mathsf{P},\ \langle \boldsymbol{x}, \boldsymbol{y}\rangle \le 1\} = \{\boldsymbol{y} \in \mathbb{R}^d \ ; \ \forall i,\ \langle \boldsymbol{v}_i, \boldsymbol{y}\rangle \le 1\}$$

⋆ **Exercise 1.2.21**
What is the polar of the regular n-gon?
What is the polar of the standard d-simplex?
What is the polar of the standard d-cube? Of the standard d-cross-polytope?

⋆ **Exercise 1.2.22**
For a face F of P with $\boldsymbol{0} \in \mathrm{int}(\mathsf{P})$, consider $\mathsf{F}^\diamond := \left\{ \boldsymbol{y} \in \mathbb{R}^d \ ; \ \begin{array}{l} \forall \boldsymbol{x} \in \mathsf{P},\ \langle \boldsymbol{x}, \boldsymbol{y}\rangle \le 1 \\ \forall \boldsymbol{x} \in \mathsf{F},\ \langle \boldsymbol{x}, \boldsymbol{y}\rangle = 1 \end{array} \right\}$. Show that $\mathsf{F}^\diamond$ is a face of P°.

Show that $\left(\mathsf{P}^\circ\right)^\circ = \mathsf{P}$, and $\left(\mathsf{F}^\diamond\right)^\diamond = \mathsf{F}$.
Show that $\mathscr{L}_{\mathsf{P}^\circ} \simeq \mathscr{L}_{\mathsf{P}}^{\mathrm{opp}} = (\mathcal{F}(\mathsf{P}), \supseteq)$.
Using Exercise 1.2.18, conclude that $\mathscr{L}_{\mathsf{P}}$ is a lattice.

Conjecture 1.2.23 — [Kalai's cube-simplex conjecture]
For every $k \ge 1$, there exists a dimension d such that every polytope of dimension $\ge d$ has a k-face which is either a k-simplex or a k-cube.

Definition 1.2.24 — [f-vector]
We denote the set of faces of dimension k of P by $\mathcal{F}_k(\mathsf{P})$.
The f-vector of the polytope P is $f_{\mathsf{P}} := (f_{-1}, f_0, f_1, \dots, f_{\dim \mathsf{P}})$.
The f-polynomial is $F_{\mathsf{P}}(X) = \sum_{i=0}^d f_i X^i$.
The vertex set of P is $V(\mathsf{P}) := \mathcal{F}_0(\mathsf{P})$.
The edge set of P is $E(\mathsf{P}) := \mathcal{F}_1(\mathsf{P})$.

Remark 1.2.25 There are several definition of the f-polynomial, depending on what the author wants to do with it. Namely:
- $\overline{F}_{\mathsf{P}}(X) = \sum_{i=0}^{d+1} f_{i-1} X^i$
- $\tilde{F}_{\mathsf{P}}(X) = \sum_{i=0}^{d} f_{i-1} X^i$
- $\hat{F}_{\mathsf{P}}(X) = \sum_{i=0}^{d} f_{i-1} X^{d-i}$

⋆ **Exercise 1.2.26**
Compute the f-polynomial of a d-simplex $\overline{F}_{\Delta_{d-1}}(X)$, a d-cube $F_{\square_d}(X)$, a d-cross-polytope with $\tilde{F}_{\diamond_d}(X)$, a n-gon.

⋆ **Exercise 1.2.27**
Express $\overline{F}_{\mathsf{P}^\circ}$ in term of $\overline{F}_{\mathsf{P}}$.

Conjecture 1.2.28 — [Kalai's 3^d conjecture]
If P is a centrally symmetric d-polytope (i.e. $\mathsf{P} = -\mathsf{P}$), then $F(1) = \sum_i f_i \ge 3^d$, and the equality cases are exactly Hanner polytopes.

Definition 1.2.29 — [h-vector, h-polynomial]
The h-vector of a d-polytope P *is the vector* $h_{\mathsf{P}} = (h_0, \dots, h_d)$ *whose associated h-polynomial* $H_{\mathsf{P}}(X) = \sum_{j=0}^{d} h_j X^j$ *is defined by the polynomial relation:*

$$H_{\mathsf{P}}(X) = F_{\mathsf{P}}(X-1)$$

Remark 1.2.30 As for the f-polynomial, the h-vector and h-polynomial may be defined differently in other courses/papers, depending on what the authors aim at. In particular, for Stanley–Reisner ring, Hilbert–Poincaré series, and Ehrhart theory, the following definition is more appropriate: $\sum_{i=0}^{\dim \mathsf{P}} f_{i-1}(X-1)^{\dim \mathsf{P}-i} = \sum_{j=0}^{\dim \mathsf{P}} h_j X^{\dim \mathsf{P}-j}$.

⋆ **Exercise 1.2.31**
Unravel this polynomial relation to write explicitly the relation between the h_j and the f_i.

◊ **Construction 1.2.32**
In order to compute the h-vector from the f-vector, there is a very simple and graphic way. Let's compute the one of the 3-cube: the f-vector is $(8, 12, 6, 1)$ (we dropped $f_{-1} = 1$ on purpose). We write it (in reverse) on the right side of a Pascal-like triangle, against a line of 1s, and compute iterated subtractions, as follows.

1	1	1
1 6	1 8	1 8
1 **6-1** 12	1 5 12	1 5 12
1 8	1 **5-1** **12-5** 6	1 4 7 6
1	1	**1** **3** **3** **1**

As a result, we get: $h_{\square_3} = (1,3,3,1)$. Recall that $F_{\square_3}(X) = (X+2)^3$, hence $H_{\square_3}(X) = F_{\square_3}(X-1) = (X+1)^3$, which agrees with our result.

Section 1.3

Simple polytopes, simplicial polytopes, in-degree vectors

As we use "vertices" and "edges" for polytope, we (try to) use "nodes" and "arcs" for graphs. There remains surely some misuses of words in this lecture notes: the reader is invited to point them out.

Definition 1.3.1 — [Linear program]
(In this course,) a (bounded) linear program $(\mathsf{P}, \boldsymbol{c})$ *is a couple formed by a polytope* $\mathsf{P} \subset \mathbb{R}^d$*, and a direction* $\boldsymbol{c} \in \mathbb{R}^d$*.*

We will properly study linear programs in a later course.

Definition 1.3.2 — [(Directed) graph of a polytope]
The graph of a polytope $\mathsf{P} \subset \mathbb{R}^d$ *is the graph* G_{P} *whose node set is the vertex set* $V(\mathsf{P})$*, and whose arc set is the edge set* $E(\mathsf{P})$*.*

Once chosen a direction $\boldsymbol{c} \in \mathbb{R}^d$*, one can define the directed graph of a linear program* $(\mathsf{P}, \boldsymbol{c})$ *as the directed graph* $G_{\mathsf{P},\boldsymbol{c}}$ *whose node set is* $V(\mathsf{P})$ *and where there is an arc* $\boldsymbol{u} \to \boldsymbol{v}$ *if* $\boldsymbol{uv} \in E(\mathsf{P})$ *and* $\langle \boldsymbol{u}, \boldsymbol{c}\rangle < \langle \boldsymbol{v}, \boldsymbol{c}\rangle$*. The set of directed arcs of* $G_{\mathsf{P},\boldsymbol{c}}$ *is denoted* $E_{\boldsymbol{c}}(\mathsf{P})$*.*

A direction $\boldsymbol{c} \in \mathbb{R}^d$ *is called generic when* $\langle \boldsymbol{u}, \boldsymbol{c}\rangle \neq \langle \boldsymbol{v}, \boldsymbol{c}\rangle$ *for all edge* $\boldsymbol{uv} \in E(\mathsf{P})$*, that is to say when* $G_{\mathsf{P},\boldsymbol{c}}$ *is an orientation of* G_{P}*.*

⋆ **Exercise 1.3.3**
Prove that generic directions always exists.

What is the minimal degree of a node in G_{P}.

Definition 1.3.4 — [Simple, simplicial]
In a d-polytope, a vertex $\boldsymbol{v} \in V(\mathsf{P})$ *is simple if* $\boldsymbol{v}$ *is contained in exactly* d *edges of* P*. A d-polytope is simple if all its vertices are simple, i.e. if* G_{P} *is d-regular.*

A polytope is simplicial if all its facets are simplices.

⋆ **Exercise 1.3.5**
Among simplices, cubes, cross-polytopes, hypersimplices, and polygons, which are simple, which are simplicial?

⋆ **Exercise 1.3.6**
Let $\boldsymbol{v} \in V(\mathsf{P})$ be a simple vertex. Show that $\mathsf{P}/\boldsymbol{v}$ is a simplex.

Deduce that for any subset of edges adjacent to $\boldsymbol{v}$ of cardinal k, there exists a k-face of P that contains these edges.

Deduce also that the dual of a simple polytope is a simplicial polytope, and reciprocally.

⋆ **Exercise 1.3.7**
Which polytopes are both simple and simplicial?

⋆ **Exercise 1.3.8**
Show that if $\boldsymbol{c}$ is generic, then $G_{\mathsf{P},\boldsymbol{c}}$ has a unique source and a unique sink. Moreover, if $\boldsymbol{c}$ is generic, show that for every face F of P, then the restriction of $G_{\mathsf{P},\boldsymbol{c}}$ on the vertices of F also has a unique source and a unique sink.

Definition 1.3.9 — [In-degree vector]
The in-degree of a vertex $\boldsymbol{v} \in V(\mathsf{P})$ *in a linear program* $(\mathsf{P}, \boldsymbol{c})$ *is its in-degree in* $G_{\mathsf{P},\boldsymbol{c}}$*, i.e.* $in_{\mathsf{P},\boldsymbol{c}}(\boldsymbol{v}) = |\{\boldsymbol{u} \in V(\mathsf{P}) \; ; \; \boldsymbol{u} \to \boldsymbol{v} \in E_{\boldsymbol{c}}(\mathsf{P})\}|$*. The in-degree vector of a linear program* $(\mathsf{P}, \boldsymbol{c})$ *is the in-degree vector of the directed graph* $G_{\mathsf{P},\boldsymbol{c}}$*, that is to say the vector* $in_{\mathsf{P},\boldsymbol{c}}$ *whose j-th coordinate is* $\big|\{\boldsymbol{v} \in V(\mathsf{P}) \; ; \; in_{\mathsf{P},\boldsymbol{c}}(\boldsymbol{v}) = j\}\big|$*.*

The out-degree and out-degree vector are defined in the same manner.

Theorem 1.3.10 *For a simple polytope* P *and any generic direction* $\boldsymbol{c}$*, one has:* $in_{\mathsf{P},\boldsymbol{c}} = h_{\mathsf{P}}$

⋆ **Exercise 1.3.11**
Think about the version for simplicial polytopes.

⋆ **Exercise 1.3.12**
Fix a linear program $(\mathsf{P}, \boldsymbol{c})$ where P is simple and $\boldsymbol{c}$ generic. Using Exercises 1.3.6 and 1.3.8, double-count the couples $(\boldsymbol{v}, \mathsf{F})$ where $\dim \mathsf{F} = i$, and $\boldsymbol{v} \in V(\mathsf{F})$ is a sink of F in $G_{\mathsf{P},\boldsymbol{c}}$.

Deduce Theorem 1.3.10.

(Semi-)open problem 1.3.13 *Characterize the polytopes* P *such that there is a generic direction* $\boldsymbol{c}$ *with* $in_{\mathsf{P},\boldsymbol{c}} = h_{\mathsf{P}}$*.*

Positive example, see: Exercise 2.4.15 (I have other examples if you want.)

Section 1.4

Dehn–Sommerville relations, Euler characteristic

Theorem 1.4.1 — [Dehn 1905 & Sommerville 1927]
For a simple d-polytope P, one has $h_i = h_{d-i}$.

⋆ Exercise 1.4.2
Pick $\boldsymbol{c}$ generic and consider the in-degree vectors for the linear programs $(\mathsf{P}, \boldsymbol{c})$ and $(\mathsf{P}, -\boldsymbol{c})$. Think about it long enough and deduce the theorem.

Write the statement in terms of f-vectors.

Think about writing the counterpart for simplicial polytopes.

Corollary 1.4.3 *Dehn–Sommerville relations are the only (linear) relations among the f-vectors of simple d-polytopes.*

⋆ Exercise 1.4.4
(Wait for the definition of cyclic polytopes in Section 1.5.)

Show that the f-vectors of $\mathsf{Cyc}(d, n)^\circ$ are linearly independent for $d + 1 \leq n \leq d + \lfloor \frac{d}{2} \rfloor + 1$. Deduce Corollary 1.4.3.

Definition 1.4.5 — [Euler's characteristic]
The Euler's characteristic of a d-polytope (non-necessarily simple) is $\chi(\mathsf{P}) = \sum_{i=0}^{d} (-1)^i f_i = F_{\mathsf{P}}(-1)$.

⋆ Exercise 1.4.6
Using $F(X - 1) = H(X)$ and Theorem 1.4.1, prove that, for a simple polytope $\chi(\mathsf{P}) = 1$.

Theorem 1.4.7 — [Euler relation]
For all polytopes, $\chi(\mathsf{P}) = 1$.

⋆ Exercise 1.4.8
See Section 2.F.

Corollary 1.4.9 *The relation $\chi(\mathsf{P}) = 1$ is the only (linear) equation satisfied by all f-vectors of d-polytopes.*

⋆ Exercise 1.4.10
A *pyramid* $\mathsf{Pyr}(\mathsf{P})$ over a polytope P is a polytope whose face lattice is isomorphic to the one of $\operatorname{conv}\{(\boldsymbol{v}, 0) \,;\, \boldsymbol{v} \in V(\mathsf{P})\} \cup \{(\boldsymbol{0}, 1)\}$.

Compute $\overline{F}_{\mathsf{Pyr}(\mathsf{P})}(X)$ in term of $\overline{F}_{\mathsf{P}}(X)$.

Let $\mathsf{Pyr}^i(\mathsf{P})$ be defined by $\mathsf{Pyr}^{i+1}(\mathsf{P}) = \mathsf{Pyr}(\mathsf{Pyr}^i(\mathsf{P}))$ and $\mathsf{Pyr}^0(\mathsf{P}) = \mathsf{P}$. For a fixed dimension d, pick $\mathsf{P}_i = \mathsf{Pyr}^i(\square_{d-i})$. Computing $\overline{F}_{\mathsf{P}_i}(X)$, show that the f-vectors of P_i are independent, and deduce Corollary 1.4.9.

Section 1.5

Cyclic polytopes, Upper bound theorem

We have seen that the f-vector respect some equalities, especially when P is simple (or simplicial). But what about inequalities?

Question: For each $i \in [d]$, can one give bounds on $f_{\mathsf{P},i}$ in term of d and $f_{\mathsf{P},0}$, for all d-polytopes?

Definition 1.5.1 — [Cyclic polytope, moment curve]
For $d \geq 1$, the moment curve is the curve defined by the function $\gamma_d : t \mapsto (t, t^2, \dots, t^d)$.
For $\boldsymbol{t} = t_1, \dots, t_n$, the cyclic polytope associated to $\boldsymbol{t}$ is $\mathsf{Cyc}(d, \boldsymbol{t}) := \operatorname{conv}\{\gamma_d(t_i) \ ; \ i \in [n]\}$.

⋆ Exercise 1.5.2
A polytope P is said to be *k-neighborly* when all $\operatorname{conv}\{\boldsymbol{v} \in X\}$ for all $X \subseteq V(\mathsf{P})$ with $|X| \leq k$ is a face of P. A polytope is said to be *neighborly* when it is $\lfloor \frac{d}{2} \rfloor$-neighborly.

Show that if P is k-neighborly, then all its j-faces for $j \leq k$ are simplices.

Find a neighborly polytope. Comment on how weird it is for a polytope to be neighborly.

⋆ Exercise 1.5.3
Show that the facets of $\mathsf{Cyc}(d, \boldsymbol{t})$ are simplices (Hint: Use a clever polynomial).

Show that a cyclic polytope $\mathsf{Cyc}(d, \boldsymbol{t})$ is neighborly (Hint: Use a clever polynomial).

Show that $\mathsf{Cyc}(d, \boldsymbol{t})^\circ$ is simple, and $h_i(\mathsf{Cyc}(d, \boldsymbol{t})^\circ) = \binom{n-d+i-1}{i}$ for $i \leq \lfloor \frac{d}{2} \rfloor$ and $h_i(\mathsf{Cyc}(d, \boldsymbol{t})^\circ) = \binom{n-i-1}{d-i}$ for $i > \lfloor \frac{d}{2} \rfloor$.

Theorem 1.5.4 *For all $\boldsymbol{t} \in \mathbb{R}^n$, the cyclic polytopes $\mathsf{Cyc}(d, \boldsymbol{t})$ have the same face lattice.*
Consequently, when we only care about face lattice, we denote $\mathsf{Cyc}(d, n)$ a polytope such that there exists $\boldsymbol{t} \in \mathbb{R}^n$ with $\mathsf{Cyc}(d, n) = \mathsf{Cyc}(d, \boldsymbol{t})$.

⋆ Exercise 1.5.5 — [Gale evenness criterion]
Fix $\boldsymbol{t} = (t_1, \dots, t_n) \in \mathbb{R}^n$. To a subset $X \subseteq [n]$, associate $\mathsf{F}_X = \operatorname{conv}\{\gamma_d(t_i) \ ; \ i \in X\}$. Prove that F_X forms a facet of $\mathsf{Cyc}(n, \boldsymbol{t})$ if and only if $|X| = d$ and any two elements in $[n] \smallsetminus X$ are separated by an even number of elements from $[n]$ in the sequence $(t_1, \dots, t_n)$.

Deduce the previous theorem.

(Semi-)open problem 1.5.6 *Find $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ with the smallest (positive) integer coordinates such that $\operatorname{conv}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ have the same face lattice as a cyclic polytope.*

Theorem 1.5.7 — [Upper bound theorem – McMullen 1970 & Stanley 1975]
For all d-polytopes P with n vertices, one has $f_i(\mathsf{P}) \leq f_i(\mathsf{Cyc}(d, n))$.

⋆ Exercise 1.5.8
Prove that, for any d-polytope with n vertices, we have $f_i \leq \binom{n}{i+1}$.

⋆ Exercise 1.5.9
Fix a d-polytope P. By slightly modifying the coordinates of the vertices of P, show that there exists a polytope $\tilde{\mathsf{P}}$ that is simplicial, with $f_i(\mathsf{P}) \leq f_i(\tilde{\mathsf{P}})$.

Deduce that it is enough to prove $h_i(\mathsf{P}) \leq h_i(\mathsf{Cyc}(d, n)^\circ)$ for P a simple d-polytope with n **facets**.

Fix a simple d-polytope with n facets. By considering the vertices with in-degree i and the ones with in-degree $i+1$, show that:

$$(d-i)h_i(\mathsf{P}) + (i+1)h_{i+1}(\mathsf{P}) = \sum_{\mathsf{F} \text{ facets of } \mathsf{P}} h_i(\mathsf{F})$$

where $d - i = \binom{d-i}{d-i-1}$ and $i + 1 = \binom{i+1}{i}$.

Convince yourself that $h_i(\mathsf{F}) \leq h_i(\mathsf{P})$, and deduce that $h_{i+1}(\mathsf{P}) \leq \frac{n-d+i}{i+1} h_i(\mathsf{P})$.

Deduce that $h_i(\mathsf{P}) \leq \binom{n-d+i-1}{i}$ for $i \leq \lfloor \frac{d}{2} \rfloor$ and $h_i(\mathsf{P}) \leq \binom{n-i-1}{d-i}$ for $i > \lfloor \frac{d}{2} \rfloor$.

Conclude.

(Semi-)open problem 1.5.10 *For interesting classes of polytopes (e.g. lattice polytopes, centrally symmetric, rotational invariant, etc), state an upper bound theorem.*

Section 1.6

Stacked polytopes, Lower bound theorem

Definition 1.6.1 — [Vertex stacking, stacked polytope]
For a polytope P *and a facet* F*, a vertex stacking of* F *in* P *is a polytope* $\mathrm{conv}(\mathsf{P} \cup \{\boldsymbol{x}\})$ *where* $\boldsymbol{x}$ *(strictly) satisfies all the inequalities that define* P *but does not satisfy the inequality defining* F*.*

A n*-stacked polytope* $\Delta_{d-1}^{\uparrow n-d}$ *is a polytope obtained by stacking* $n-d$ *times on a* d*-simplex.*

⋆ **Exercise 1.6.2**
Compute the f-vector of a stacking in term of the vector of P.

Prove that a n-stacked polytope is simplicial, and compute its f-vector (and show this independently from the chosen sequence of stackings). In particular, show it has n vertices.

Theorem 1.6.3 — [Lower bound theorem – Barnette 1973]
For all simplicial d*-polytopes* P *with* n *vertices:*

$$f_i(\mathsf{P}) \geq f_i(\Delta_{d-1}^{\uparrow n-d})$$

Remark 1.6.4 For the proof, see Barnette's original article [Bar73].

⋆ **Exercise 1.6.5**
Using duality/polarity, write a similar lower bound theorem for simple polytopes. Especially, construct the operation which is dual to (simplicial) vertex stacking, and the dual of a n-stacked polytope.

(Semi-)open problem 1.6.6 *State a corresponding lower bound theorem for all* d*-polytopes, see [Xue20, PVTY24].*

APPENDICE 1.G

f-vector of 2-polytopes and 3-polytopes

⋆ **Exercise 1.G.1 — [f-vector of polygons]**
Characterize the set of possible f-vectors of polygons.

Show that each possible f-vector correspond to a unique face lattice.

⋆ **Exercise 1.G.2 — [f-vector of 3-polytopes]**
Let P be a 3-polytopes, show that its f-vector can be recovered from f_0 and f_2 only.

In the plane, draw two orthogonal axes, the horizontal being labeled f_0 and the vertical f_2. Each axis goes from 0 to 10. Place on this graphic the f-vector of a 3-simplex, a 3-cube, a 3-cross-polytope (*a.k.a.* an octahedron). Place the f-vectors of pyramids over n-gons, of prism over n-gons.

At most, how many edges can be adjacent to a vertex of a 3-polytopes? Deduce that $f_0 \leq 2f_2 - 4$, and that the equality is satisfied by simple 3-polytopes. Draw the line $f_0 = 2f_2 - 4$ on your graphic.

Dually, at least of how many vertices shall a facet of a 3-polytope contain? Deduce that $f_2 \leq 2f_0 - 4$, and that the equality is satisfied by simplicial 3-polytopes. Draw the line $f_2 = 2f_0 - 4$ on your graphic.

You now know that the f-vector of a 3-polytope correspond to an integer point of the cone C you have drawn (defined by $f_0 \leq 2f_2 - 4$ and $f_2 \leq 2f_0 - 4$). It remains to prove the reciprocal, *i.e.* for each integer point of the cone C, there exists a polytope whose f-vector correspond to this point.

Interpret $\mathsf{P} \mapsto \mathsf{P}^\circ$ on your graphic.

Stack a vertex on the facet of a simplicial 3-polytope, what happens to its f-vector?

For a polytope P and a vertex $\boldsymbol{v} \in V(\mathsf{P})$, a *vertex truncation* of $\boldsymbol{v}$ in P is a polytope defined by intersecting P with $H^+_{\boldsymbol{a},b-\varepsilon}$ for $H_{\boldsymbol{a},b}$ a supporting hyperplane of $\boldsymbol{v}$. Equivalently, a vertex truncation is $\mathrm{conv}\big(\mathsf{P} \smallsetminus \boldsymbol{v}\big) \cup \{\boldsymbol{w} \in v(\mathsf{P})\ ;\ \boldsymbol{w} \neq \boldsymbol{v}\}$.

By starting with a 3-cross-polytope, then cleverly doing a series of vertex stacking, then doing a clever vertex truncation, construct a 3-polytope for each integer point in the cone C. This proves the following Theorem 1.G.3.

Find two 3-polytopes with different face lattices but the same f-vector.

Theorem 1.G.3 — [f-vector of 3-polytopes]
A vector $(1, f_0, f_1, f_2, 1)$ is the f-vector of a 3-polytope, if and only if:

$$f_0 - f_1 + f_2 = 2 \quad , \quad f_0 \leq 2f_2 - 4 \quad \text{and} \quad f_2 \leq 2f_0 - 4$$

(Semi-)open problem 1.G.4 *Characterize the set of f-vectors of 4-polytopes.*

(Semi-)open problem 1.G.5 — [Fatness]
The fatness of a 4-polytope P is: $\mathrm{fat}(\mathsf{P}) := \frac{f_1+f_2}{f_0+f_3}$. Does there exists a number $c \in \mathbb{R}$ such that $\mathrm{fat}(\mathsf{P}) \leq c$ for all 4-polytope?

⋆ **Exercise 1.G.6 — [Number of edges]**
Where is the set of f-vector corresponding to 3-polytopes with k edges for a given k.

Show that $F_1 := \{f_1\ ;\ (1, f_0, f_1, f_2, 1)$ the f-vector of a 3-polytope$\}$ is not (the integral points of a) convex set, by looking at 3-polytopes with 7 edges.

Compute F_1 explicitly.

(Semi-)open problem 1.G.7 *For each (f_0, f_2), one can compute the number of (non-isomorphic) face lattices with this f-vector, denote by $L(f_0, f_2)$ this number. One can conjecture two things: (a) $L(f_0, f_2)$ increases when $f_0 + f_2$ increases; (b) $L(f_0, k - f_0)$ is unimodal (or more) for fixed k when f_0 varies.*

In order to convince yourself of this idea, try to color your graphic with a color chart that embodies the number of non-isomorphic face lattices for each point (f_0, f_2).

CHAPTER

2

PERMUTAHEDRON, NORMAL FANS, MINKOWSKI SUMS, DEFORMATIONS

Content of the chapter

Section 2.1

Normal fans, Permutahedron

Attention 2.1.1 *Recall that a face of a polytope* $\mathsf{P} \subset \mathbb{R}^d$ *is* $\mathsf{P}^{\boldsymbol{c}} = \{\boldsymbol{x} \in \mathsf{P} \ ; \ \langle \boldsymbol{x}, \boldsymbol{c} \rangle = \max_{\boldsymbol{y} \in \mathsf{P}} \langle \boldsymbol{y}, \boldsymbol{c} \rangle\}$ *for* $\boldsymbol{c} \in \mathbb{R}^d$.

Definition 2.1.2 — [Fan]
A fan $\mathcal{F}$ *is a collection of cones* $\mathcal{F} = \big(\mathsf{C}_1, \dots, \mathsf{C}_n\big)$ *where* $\mathsf{C}_i \subseteq \mathbb{R}^d$*, such that (a) if* $\mathsf{C} \in \mathcal{F}$*, then every face of* C *is also in* $\mathcal{F}$*; (b) for all* $\mathsf{C}, \mathsf{C}' \in \mathcal{F}$*, the cone* $\mathsf{C} \cap \mathsf{C}'$ *is a face of both* C *and* C'*. The set of maximal cones (maximal for containment) of* $\mathcal{F}$ *is denoted* $\hat{\mathcal{F}}$.

A fan can be:

- *essential if there is no sub-space that is common to all the cones, i.e.* $\bigcap_{\varnothing \neq \mathsf{C} \in \hat{\mathcal{F}}} \mathsf{C} = \{\mathbf{0}\}$
- *complete if it covers the whole space, i.e.* $\bigcup_{\mathsf{C} \in \mathcal{F}} = \mathbb{R}^d$
- *pointed if all its cones are, i.e. the only vertex of* $\mathsf{C} \in \mathcal{F}$ *is* $\mathbf{0}$ *(i.e.* C *a only 1 bounded face)*

The face lattice $\mathscr{L}(\mathcal{F}) := (\mathcal{F}, \subseteq)$ *of a fan* $\mathcal{F}$ *is the collection of all its cones, ordered by inclusion.*

Definition 2.1.3 — [Normal cone, normal fan]
For a non-empty face F *of a polytope* P*, the (outer) normal cone of* F *in* P *is the cone* $\mathcal{N}_{\mathsf{P}}(\mathsf{F}) = \{\boldsymbol{c} \in \mathbb{R}^d \ ; \ \mathsf{P}^{\boldsymbol{c}} = \mathsf{F}\}$.

The (outer) normal fan of P *is the collection of all its normal cone:* $\mathcal{N}_{\mathsf{P}} = \Big(\mathcal{N}_{\mathsf{P}}(\mathsf{F}) \ ; \ \varnothing \neq \mathsf{F} \text{ face of } \mathsf{P}\Big)$.

As the normal cone of a facets is 1-dimensional, we call normal vector a vector of such cone.

The (inner) normal cones and (inner) normal fan are defined by taking min *instead of* max *in the definition of* $\mathsf{P}^{\boldsymbol{c}}$*, that is to say it is the central symmetric of the outer counterpart.*

⋆ **Exercise 2.1.4**
Compute $\mathcal{N}_{\mathsf{P}}(\mathsf{P})$.

Draw the normal fan of a n-gon.

What is the normal fan of a 3-cube?

⋆ **Exercise 2.1.5**
Show that the normal fan is a fan. Is it essential? complete? pointed?

⋆ **Exercise 2.1.6**
Show that the rays of the normal cone of a face G are the normal vectors of the facets F with $\mathsf{G} \subseteq \mathsf{F}$.

Show that $\mathscr{L}(\mathsf{P}) = \mathscr{L}^{\text{opp}}(\mathcal{N}_{\mathsf{P}})$.

♢ **Construction 2.1.7 — [Stereographic projection (of pointed fans)]**
The *stereographic projection* is a bijective projection from a sphere (minus a point) onto a plane: we will use it to visualize normal fans of 3- and 4-polytopes (and more generally all pointed fans). Indeed, remark that all the information contained in a normal fan $\mathcal{N}_{\mathsf{P}}$ can be retrieved from the intersection between $\mathcal{N}_{\mathsf{P}}$ and the unit sphere (it is just a matter of coning over the intersection).

The stereographic projection is constructed as follows.

Assume the center of the sphere is $\mathbf{0}$, and fix a point $\boldsymbol{p}$ on the sphere. This defines a (hyper)plane $H_{\boldsymbol{p}}$ orthogonal to the vector $\boldsymbol{p}$. Each point $\boldsymbol{x} \neq \boldsymbol{p}$ of the sphere is mapped on the (hyper)plane $H_{\boldsymbol{p}}$ by casting a ray from $\boldsymbol{p}$ that passes through $\boldsymbol{x}$: this rays intersect the (hyper)plane $H_{\boldsymbol{p}}$ at some image point $\mathrm{s}(\boldsymbol{x})$.

Show that, for a given $\boldsymbol{p}$, the map $\boldsymbol{x} \mapsto \mathrm{s}(\boldsymbol{x})$ is a bijection $\mathbb{S}^d \smallsetminus \{\boldsymbol{p}\} \to \mathbb{R}^{d-1}$.

What is the image by s of a great circle (a *great circle* is a sphere of dimension $d-1$ on $\mathbb{S}^d$ whose center is also $\mathbf{0}$)? Distinguish whether the great circle contains $\boldsymbol{p}$ or not.

⋆ **Exercise 2.1.8**
Draw the stereographic projection of $\mathcal{N}_{\square_2}, \mathcal{N}_{\Delta_2}, \mathcal{N}_{\square_3}, \mathcal{N}_{\Delta_3}, \mathcal{N}_{\diamondsuit_3}$.

Definition 2.1.9 — [Permutahedron]
The permutahedron is the best polytope. ☺

Definition 2.1.10 — [Permutahedron]
The n-permutahedron is the defined as follows:

$$\Pi_n := \mathrm{conv}\left\{ \begin{pmatrix} \sigma(1) \\ \sigma(2) \\ \vdots \\ \sigma(n) \end{pmatrix} \; ; \; \sigma \text{ permutation of } [n] \right\}$$

⋆ **Exercise 2.1.11**
Draw the 2- and 3-permutahedra, labeling each vertex by a permutations.
Draw the (stereographic projection of the) normal of Π_4. Label it cleverly.

⋆ **Exercise 2.1.12 — [Faces of Π_n, braid fan, braid arrangement]**
Show that $\dim \Pi_n = n - 1$.

Show that for each permutation σ, the point $\boldsymbol{p}_\sigma := \begin{pmatrix} \sigma^{-1}(1) \\ \sigma^{-1}(2) \\ \vdots \\ \sigma^{-1}(n) \end{pmatrix}$ is a vertex of Π_n. How many vertices does Π_n has?

An *ordered partition* is an ordered tuple $(B_1, \dots, B_r)$ of *blocks* $B_i \subseteq [n]$, $B_i \neq \varnothing$ satisfying $B_i \cap B_j = \varnothing$ for $i \neq j$, and $\bigcup_i B_i = [n]$. Show that the faces of Π_n are in 1-to-1 correspondence with ordered partitions of $[n]$. Do it by associating to each ordered partition a cone (the normal cone of the corresponding face).

Deduce that Π_n is a simple polytope.

Deduce that the normal fan of Π_n, called the *braid fan*, is induced by an hyperplane arrangement: an *hyperplane arrangement* is a collection $\mathcal{H} = (H_1, \dots, H_m)$ of (linear) hyperplanes, and the fan it *induces* is the fan whose maximal cones are (the closure of) the connected components of $\mathbb{R}^d \smallsetminus \bigcup_i H_i$.

Write down this hyperplane arrangement clearly and give it a name: the *braid arrangement*. The maximal cones of the braid arrangement are the *braid cones* $\mathsf{C}_\sigma := \{\boldsymbol{x} \in \mathbb{R}^n \; ; \; x_{\sigma^{-1}(1)} < x_{\sigma^{-1}(2)} < \dots < x_{\sigma^{-1}(n)}\}$ for σ a permutation of $[n]$.

⋆ **Exercise 2.1.13**
Two affine spaces are *parallel* if their underlying linear sub-spaces are the same. Two faces F, G of a polytope are *parallel faces* is $\mathrm{aff}(\mathsf{F})$ and $\mathrm{aff}(\mathsf{G})$ are parallel affine spaces.

Show that two faces are parallel if and only if their normal cone spans the same (linear) space. Show that two facets are parallel if and only if their normal vectors are opposite.

Characterize parallel faces of Π_n in term of their associated ordered partitions.

⋆ **Exercise 2.1.14**
The *Cartesian product* of two polytopes $\mathsf{P} \subset \mathbb{R}^d$ and $\mathsf{Q} \subset \mathbb{R}^{d'}$ is $\mathsf{P} \times \mathsf{Q} := \{(\boldsymbol{p}, \boldsymbol{q}) \; ; \; \boldsymbol{p} \in \mathsf{P}, \; \boldsymbol{q} \in \mathsf{Q}\} \subset \mathbb{R}^{d+d'}$. Show that the Cartesian product of two polytopes is a polytope.

Show that a face of Π_n of dimension k is a product $\Pi_{k_1} \times \dots \times \Pi_{k_r}$ with $k_1 + \dots + k_r = k$.

What are the possible 2-faces of Π_n?

(Semi-)open problem 2.1.15 *Let $\mathcal{P} = (\mathsf{P}_0, \mathsf{P}_1, \mathsf{P}_2, \dots)$ be a class of polytopes, one per dimension (i.e. $\dim \mathsf{P}_i = i$) such that all the faces of P_i are isomorphic to a product $\prod_{j_1,\dots,j_r} \mathsf{P}_{j_1} \times \dots \times \mathsf{P}_{j_r}$ where $j_1, \dots, j_r \leq i$.*

Find (a lot) of examples of such class $\mathcal{P}$, and detail the properties of such classes. Examples: simplex, cubes, permutahedron, associahedra.

See also Hopf monoids and generalized permutahedra [AA17].

⋆ **Exercise 2.1.16 — [Graph of Π_n, Coxeter groups]**
What are the possible edge directions of Π_n? How many edges are there? How many classes of parallelism do they form?

Show that the graph of the permutahedron is isomorphic to the graph whose vertices are the permutations of $[n]$ and where two permutations are linked if they differ by a transposition.

A *Coxeter group* is a (non-commutative) group generated by elements $r_1, \dots, r_n$ which satisfies $(r_i r_j)^{m_{ij}} = 1$ with $m_{ii} = 1$ (*i.e.* $r_i^2 = 1$), and $m_{ji} = m_{ij} \geq 2$ for some (not necessarily all) couples $i, j \leq n$. Such a group is characterized by the symmetric $n \times n$-matrix $M = [m_{ij}]_{i,j}$.

Show that for $M = \begin{pmatrix} 1 & 3 & \dots & 3 \\ 3 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 3 \\ 3 & \dots & 3 & 1 \end{pmatrix}$, the associated Coxeter group is the group $\mathcal{S}_n$ of permutations of $[n]$ (Hint: associate each r_i with a clever transposition).

The *Cayley graph* of a group (given via a set of generators R) is the directed graph (possibly infinite) whose vertices are the element of the group and where there is an edge from g to g' if there exists $r \in R$ satisfying $g' = g\,r$.

For $\boldsymbol{c} = (1, 2, \ldots, n)$, show that $G_{\Pi_n,\boldsymbol{c}}$ is the Cayley graph of $\mathcal{S}_n$ seen as a Coxeter group.

Characterize 2-faces of Π_n, and show that they correspond to the minimal relations between the generating elements of $\mathcal{S}_n$.

Remark 2.1.17 As you may imagine, it is interesting to look at other Coxeter groups, given by other matrices. In particular, $\mathcal{S}_n$, seen as above as a Coxeter group, is often called "the Coxeter group of type A_n". The permutahedron we are studying is hence called "permutahedron of type A", and so on. More often than not, in conference on related subject, you will hear someone ask at the end of a presentation: "And what about other types?", "Does your result also works in type B? In type D? And in exceptional types?".

One should make a complete course on Coxeter groups at some point... ☺

(Semi-)open problem 2.1.18 *Which Cayley graph is the graph of a polytope? What does the polytope tells about the properties of the underlying group?*

⋆ Exercise 2.1.19 — [Graph of Π_n, weak (Bruhat) order]

Let $\boldsymbol{c} = (1, 2, \ldots, n)$. Show that $\boldsymbol{c}$ is a generic direction for Π_n. Consider $G_{\Pi_n,\boldsymbol{c}}$ as the Hasse diagram of a poset, called the *weak Bruhat order*.

The *inversion set* of a permutation $\sigma \in \mathcal{S}_n$ is the set of couples $\mathrm{Inv}(\sigma) = \big\{(i, j)\ ;\ 1 \leq i < j \leq n,\ \sigma(i) > \sigma(j)\big\}$.

A set of couple is couples X is *transitive* when if $(i, j), (j, k) \in X$, then also $(i, k) \in X$. Show that a set of couples $X \subseteq \binom{[n]}{2}$ is an inversion set of a permutation if and only if X and its complement $\binom{[n]}{2} \smallsetminus X$ are transitive.

Using inversion sets, show that the weak order is a graded lattice.

(Semi-)open problem 2.1.20 *Which lattice is the graph of a polytope? What does the polytope tells about the properties of the lattice? (see Quotientopes)*

Section 2.2

Equivalences

Definition 2.2.1 — [Equivalences]
Two polytopes $\mathsf{P} \subset \mathbb{R}^d$ *and* $\mathsf{Q} \subset \mathbb{R}^{d'}$ *are:*

- *Combinatorially equivalent, denoted* $\mathsf{P} \sim \mathsf{Q}$*, if they have the same face lattice, i.e.* $\mathscr{L}(\mathsf{P}) = \mathscr{L}(\mathsf{Q})$
- *Normally equivalent if they have the same normal fan, i.e.* $\mathcal{N}_{\mathsf{P}} = \mathcal{N}_{\mathsf{Q}}$.
- *Affinely equivalent if one is an affine transformation of the other, i.e. there exists an affine function* $L : \mathbb{R}^d \to \mathbb{R}^{d'}$ *such that* $L(\mathsf{P}) = \mathsf{Q}$.
- *Graphically equivalent if their graphs are the same, i.e.* $G_{\mathsf{P}} = G_{\mathsf{Q}}$.
- *Oriented matroid equivalent*[1] *if* P *and* Q *have same dimension and number of vertices, and there exists a labeling* $V(\mathsf{P}) = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ *and* $V(\mathsf{Q}) = \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$ *such that for all* $i_1, \dots, i_{d+1} \in [n]$*, one has* $\mathit{sign}\left(\det\left(\begin{pmatrix}1\\ \boldsymbol{v}_{i_1}\end{pmatrix}, \dots, \begin{pmatrix}1\\ \boldsymbol{v}_{i_{d+1}}\end{pmatrix}\right)\right) = \mathit{sign}\left(\det\left(\begin{pmatrix}1\\ \boldsymbol{w}_{i_1}\end{pmatrix}, \dots, \begin{pmatrix}1\\ \boldsymbol{w}_{i_{d+1}}\end{pmatrix}\right)\right)$.

[1] We mention here for math culture, we will not discuss it for now.

Remark 2.2.2 You now understand what we mean by "a" simplex, "a" cube, and so on: a polytope combinatorially equivalent to a simplex, to a cube...

⋆ Exercise 2.2.3
Which equivalences imply the combinatorial equivalence? Which ones are implied by it?
Find two polytopes that are normally but not affinely equivalent, and conversely.

⋆ Exercise 2.2.4 — [Graph equivalent but not combinatorially equivalent]
A bi-pyramid over a polytope P is (a polytope combinatorially equivalent to) $\mathsf{BiPyr}(\mathsf{P}) := \operatorname{conv}\Big((\mathsf{P} \times \{0\}) \cup \{(\mathbf{0}, 1), (\mathbf{0}, -1)\}\Big)$.
Understand the graph of a bi-pyramid in term G_{P}. Show that the graph of $\mathsf{BiPyr}(\Delta_3)$ and $\mathsf{Pyr}(\mathsf{BiPyr}(\Delta_2))$ are the same, but that they are combinatorially different.

Remark 2.2.5 Why didn't we show an counter-example in dimension 3? or with simpler polytopes? We will see in Chapter 3 that (a) if P and Q are 3-polytopes with the same graph, then they are combinatorially equivalent; (b) if P and Q are (almost) simple polytopes with the same graph, then they are combinatorially equivalent.

(Semi-)open problem 2.2.6 *Characterize all polytopes* P *such that if* $G_{\mathsf{Q}} = G_{\mathsf{P}}$*, then* $\mathsf{Q} \sim \mathsf{P}$.

⋆ Exercise 2.2.7

Fix $\boldsymbol{v} = (v_1, \dots, v_n)$ with $v_i \neq v_j$. Prove that Π_n and $\Pi(\boldsymbol{v}) := \operatorname{conv}\left\{\begin{pmatrix}v_{\sigma(1)}\\ \vdots \\ v_{\sigma(n)}\end{pmatrix} \; ; \; \sigma \text{ permutation of } [n]\right\}$ are normally equivalent.

(Semi-)open problem 2.2.8 *Problem posed by Martin Winter, see his presentation at Santander, January 2024.*

A polytope P *is centrally symmetric if* $-\mathsf{P} = \mathsf{P}$ *where* $-\mathsf{P} := \{-\boldsymbol{x} \; ; \; \boldsymbol{x} \in \mathsf{P}\}$*, obviously.*

The coordinate symmetry s_i *is the symmetry against the hyperplane* $H_{\boldsymbol{e}_i,0}$*, i.e.* $s_i(\boldsymbol{e}_j) = (-1)^{i=j}\boldsymbol{e}_j$*. An orthant is either* $\mathbb{R}^d_+$ *(called the positive orthant), or obtained from* $\mathbb{R}^d_+$ *by a sequence of coordinate symmetries. Let* $\mathbb{O} \subset \mathbb{R}^d$ *be an orthant, and define* $\mathsf{U}_{\mathbb{O}}(\mathsf{P})$ *to be the union of* $\mathsf{P} \cap \mathbb{O}$ *with all its images obtained by a sequence of coordinate symmetries. A polytope* P *is locally coordinate symmetric if* $\mathsf{U}_{\mathbb{O}}(\mathsf{P})$ *is convex for all orthant* $\mathbb{O} \subset \mathbb{R}^d$.

Find a centrally symmetric polytope which is not combinatorially equivalent to a locally coordinate symmetric polytope.

Section 2.3

Minkowski sums, Minkowski summands

Definition 2.3.1 — [Minkowski sum]
The Minkowski sum of two polytopes $\mathsf{P}, \mathsf{Q} \subset \mathbb{R}^d$ *is:*

$$\mathsf{P} + \mathsf{Q} := \{\boldsymbol{p} + \boldsymbol{q} \; ; \; \boldsymbol{p} \in \mathsf{P}, \, \boldsymbol{q} \in \mathsf{Q}\}$$

⋆ **Exercise 2.3.2**
Prove that $\mathsf{P}+\mathsf{Q}$ is a projection of $\mathsf{P} \times \mathsf{Q}$.

Prove that all the vertices of $\mathsf{P}+\mathsf{Q}$ are sums of a vertex of P and a vertex of Q. Find an example where the reciprocal does not hold.

Definition 2.3.3 — [Common refinement of fans]
The common refinement of two fans $\mathcal{F}$, $\mathcal{F}'$ *is the fan whose cones are* $\mathsf{C} \cap \mathsf{C}'$ *for* $\mathsf{C} \in \mathcal{F}$ *and* $\mathsf{C}' \in \mathcal{F}'$.

⋆ **Exercise 2.3.4**
Prove that the common refinement of two fans is a fan.

Theorem 2.3.5 *The normal fan* $\mathcal{N}_{\mathsf{P}+\mathsf{Q}}$ *is the common refinement of the normal fans* $\mathcal{N}_{\mathsf{P}}$ *and* $\mathcal{N}_{\mathsf{Q}}$.

Definition 2.3.6 — [Minkowski summands, deformations]
A deformation, a.k.a. Minkowski summand of a polytope P *is a polytope* Q *such that there exists a polytope* R *and a dilation factor* $\lambda > 0$ *satisfying* $\mathsf{P} = \lambda\mathsf{Q} + \mathsf{R}$.

(Semi-)open problem 2.3.7 *Pick your favorite polytope, compute all its deformations (for almost any polytope, this has not been done yet). Hint: wait for the proper course on deformations and indecomposability before doing so.* ☺

Definition 2.3.8 — [Coarsening of fans]
A complete fan $\mathcal{G}$ *coarsens a complete fan* $\mathcal{F}$, *denoted* $\mathcal{G} \trianglelefteq \mathcal{F}$, *if every cone of* $\mathcal{F}$ *is included in a cone of* $\mathcal{G}$.

⋆ **Exercise 2.3.9**
Prove that $\mathcal{G}$ coarsens $\mathcal{F}$ if and only if every cone of $\mathcal{G}$ is a union of cones of $\mathcal{F}$.

Theorem 2.3.10 *A polytope* Q *is a deformation of a polytope* P *if and only if* $\mathcal{N}_{\mathsf{Q}} \trianglelefteq \mathcal{N}_{\mathsf{P}}$. *Hence, we denote this by* $\mathsf{Q} \trianglelefteq \mathsf{P}$.

⋆ **Exercise 2.3.11**
Prove this theorem.

(Semi-)open problem 2.3.12 *A fan is polytopal when it is the normal fan of a polytope. Fix a polytopal fan* $\mathcal{F}$, *and let* $\mathcal{F}'$ *be the fan obtained as the intersection of* $\mathcal{F}$ *by some (linear) sub-space* L. *Suppose* $\mathcal{G}'$ *is a polytopal fan which coarsens* $\mathcal{F}'$. *Under which conditions does there exists a polytopal fan* $\mathcal{G}$, *coarsening* $\mathcal{F}$, *such that* $\mathcal{G}'$ *is the intersection of* $\mathcal{G}$ *by* L*?*

⋆ **Exercise 2.3.13**
Let Q, Q' be deformations of P, and $\lambda > 0$, $\boldsymbol{t} \in \mathbb{R}^d$. Show that $\lambda\mathsf{Q} \trianglelefteq \mathsf{P}$, $\mathsf{Q} + \boldsymbol{t} \trianglelefteq \mathsf{P}$, and $\mathsf{Q} + \mathsf{Q}' \trianglelefteq \mathsf{P}$.

Prove that $\trianglelefteq$ is an order relation on the deformations of P.

Definition 2.3.14 — [Lattice of deformations]
The lattice of deformations of P *is the poset whose elements are the classes of normally equivalent deformations of* P, *ordered by* $\trianglelefteq$. *Equivalently, it is the poset of fans which coarsens* $\mathcal{N}_{\mathsf{P}}$, *ordered by* $\trianglelefteq$.

Remark 2.3.15 We will not prove it is a lattice for now, wait for the proper study of the deformation cone.

⋆ **Exercise 2.3.16**
Draw the lattice of deformations of Π_3: start by drawing the braid fan in dimension 2, then progressively merge its maximal cones. Compare the result to the lattice of faces of $\mathsf{BiPyr}(\Delta_2)$.

Definition 2.3.17 — [Minkowski indecomposability]
A polytope P *is Minkowski indecomposable if all its deformations are dilations of itself, i.e. if* $\mathsf{Q} \trianglelefteq \mathsf{P}$, *then* $\mathsf{Q} = \lambda\mathsf{P}$ *for some* $\lambda \geq 0$.

⋆ **Exercise 2.3.18**
Characterize Minkowski indecomposable polytopes of dimension 1 and 2.

Show that every simplex is Minkowski indecomposable.

Remark 2.3.19 We will go back to this notions (deformations & Minkowski indecomposability) in Chapter 4.

Section 2.4

Zonotopes, Graphical Zonotopes

The aim of this section is to study the "easiest" Minkowski sums, that is to say the Minkowski sum of (finitely many) segments. Besides, we will see the richness of deformations of the permutahedron. The underlying combinatorics of the permutahedron is the one of permutations, *i.e.* of sorting a list of number when we are allowed to compare any pair of number: What happens when we are no longer allowed to compare any pair of number, but just certain pairs? Can we get insights on other combinatorial problems, like orientation of graphs, by embedding them on (deformations of) the permutahedron?

Definition 2.4.1 — [Zonotopes]
A zonotope is a Minkowski sum of segments, that is to say a polytope that can be written $\mathsf{P} := \sum[\boldsymbol{v}_i, \boldsymbol{w}_i]$ *for some* $\boldsymbol{v}_i, \boldsymbol{w}_i \in \mathbb{R}^d$.

⋆ **Exercise 2.4.2**
Prove that zonotope is centrally symmetric (with respect to its barycenter).

⋆ **Exercise 2.4.3**
Show that the cube is a zonotope.

By thinking the cube as a product, prove that zonotopes are precisely projections of a cube.

Deduce that all the faces of a zonotope are zonotopes.

Show that a 2-dimensional zonotopes is a centrally symmetric $2n$-gons.

Reciprocally, suppose that all the 2-faces of P are centrally symmetric polygons. Show that P is a zonotope (Hint: construct the equivalence relation where two edges which are opposite in a 2-face are equivalent, then "remove" a class of such edges, and use induction).

⋆ **Exercise 2.4.4**
Show that the translation of a zonotope is still a zonotope.

Show that the normal fan of a zonotope is induced by an hyperplane arrangement.

Show that Π_n is a zonotope (Hint: reverse-engineer the suitable segments from its normal fan).

Deduce that there exists deformations of zonotopes which are not zonotopes.

⋆ **Exercise 2.4.5**
Prove that every polytope is the deformation of a zonotope.

⋆ **Exercise 2.4.6**
For $V = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\} \subset \mathbb{R}^d$, we construct the induced zonotope $\mathsf{Z}(V) = \sum_i [\boldsymbol{0}, \boldsymbol{v}_i]$.

By cleverly tiling $\mathsf{Z}(V)$ into small cubes, prove that:

$$\mathrm{Vol}\big(\mathsf{Z}(V)\big) = 2^d \sum_{1 \le i_1 < \dots < i_d \le m} \big|\det(\boldsymbol{v}_{i_1}, \dots, \boldsymbol{v}_{i_m})\big|$$

Show that $\mathrm{Vol}(\Pi_n) =$ number of (labeled) trees on n nodes $= n^{n-2}$

(Semi-)open problem 2.4.7 — [White Whale]
Introduced by Billera, the White Whale is the d-dimensional zonotope $\sum_{X \subset [d]} [\boldsymbol{0}, \boldsymbol{e}_X]$. *Compute the number of vertices of the White Whale for* $d \ge 10$. *For* $d = 9$, *the White Whale has* 1 955 230 985 997 140 *vertices, see [DHP22].*

Definition 2.4.8 — [Graphical zonotope]
The graphical zonotope associated to a graph $G = (V, E)$ *is the zonotope* $\mathsf{Z}_G := \sum_{ij \in E} [\boldsymbol{e}_i, \boldsymbol{e}_j]$ *where* $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ *is the canonical basis of* $\mathbb{R}^V$.

The weighted graphical zonotope of a graph $G = (V, E)$ *endowed with a positive weight* $w : E \to \mathbb{R}_+$ *is the zonotope* $\mathsf{Z}_{G,w} := \sum_{ij \in E} w(ij)[\boldsymbol{e}_i, \boldsymbol{e}_j]$.

⋆ **Exercise 2.4.9**
Construct a zonotope which is not a (weighted) graphical zonotopes.

If Z_G has no hexagonal 2-face, what can be said of G?

If Z_G is a cube, what can be said of G?

⋆ **Exercise 2.4.10**
Show that Z_{K_n} is (a translate of) Π_n.

Show that if H is a sub-graph of G (*i.e.* $E(H) \subseteq E(G)$), then $\mathsf{Z}_{H,w}$ is a deformation of Z_G for all positive weight w. Deduce that all weighted graphical zonotopes are deformations of Π_n.

Show that if a segment S is part of the definition of a zonotope Z, then $\mathsf{S} \trianglelefteq \mathsf{Z}$.

Deduce that the only zonotopal deformations of Π_n are the weighted graphical zonotopes (up to translation), and more generally, the only zonotopal deformations of Z_G are the weighted graphical zonotopes $\mathsf{Z}_{H,w}$ for H a sub-graph of G.

⋆ Exercise 2.4.11
An *acyclic orientation* of a graph $G = (V, E)$ is a directed graph D with the same vertex set and edge sets (but oriented) such that there is no directed cycle in D. The set of acyclic orientations of G is denoted $\mathcal{O}(G)$.

Show that the vertices of Z_G are in bijection with its acyclic orientations.

An *ordered partition* of a graph $G = (V, E)$ is a couple (μ, ρ) where μ is a partition of V and $\rho \in \mathcal{O}(G/\mu)$ where G/μ is the contraction of G on the parts of μ.

Show that the faces of Z_G are in bijection with its ordered partitions, and that the face associated to (μ, ρ) is normally equivalent to $\mathsf{Z}_{G/\mu}$.

A *topological order* of an acyclic orientation $\rho \in \mathcal{O}(G)$ is an ordering $\preccurlyeq$ of the vertices (*i.e.* a permutation of V) such that if an edge is directed $u \to v$ by ρ, then $u \prec v$.

Show that for all $\rho \in \mathcal{O}(G)$, there exists (at least) one topological order, using a graph-theoretic argument.

Let $\boldsymbol{v}_\rho$ be the vertex of Z_G associated to $\rho \in \mathcal{O}(G)$. Show that the set of topological orders of $\rho \in \mathcal{O}(G)$ are in bijection with the set permutations S such that $\mathcal{N}_{\mathsf{P}}(\boldsymbol{v}_\rho) = \bigcup_{\sigma \in S} \mathsf{C}_\sigma$, where C_σ are the braid cones defined in Exercise 2.1.12.

⋆ Exercise 2.4.12 — [Happy-edge & non-leaving-face properties]
Before coming back to graphical zonotope, let's look at the following problem. A *n-bit sequence* is a $s \in \{0, 1\}^n$. A *flip* of a bit sequence consist in changing the value of 1 bit. Fix two n-bit sequences s and s': show it is possible to flip bits from s to s' while never flipping a bit on which s and s' agree, use the existence of a certain polytope to (almost) immediately get a proof.

Let's do solve similar problem on graphs. Fix a graph G. A *re-orientation* of an acyclic orientation is another acyclic orientation which differs on the orientation of a single edge.

Consider two acyclic orientations ρ and ρ'. Show that there exists a sequence of re-orientation which goes from ρ to ρ'.

Let $G_1, \dots, G_r$ be disjoint induced sub-graphs on which ρ and ρ' agrees (*i.e.* give the same orientation to the edges). Show that there exists a sequence of re-orientation which goes from ρ to ρ' and never modify the orientation of an edge in any G_i.

> **Remark 2.4.13** It is usual in graph theory of more generally in combinatorics to define a collection of "configurations", and a notion of "flip" or "adjacency" (see bases of matroids, Catalan families, permutations, *etc...*). This gives rise to a very large *graph of flips* Γ whose vertex set is the collection of configurations, and two configurations are linked if they differ by a flip. A first usual question is "Is the graph of flip Γ connected?" and a second question is "Given two configurations C_1, C_2, and $C_1 \cap C_2$ the sub-configuration shared by C_1 and C_2, are C_1 and C_2 connected using only configurations containing $C_1 \cap C_2$, *i.e.* is $\Gamma|_{C_1 \cap C_2}$ connected?". The second question is called having the *happy-edge property*.
>
> If one managed, as for binary trees, permutations, bit sequences, acyclic orientations of graphs, etc, to embed the configurations into the vertices of a polytope, and the flip into the edges of this polytope, then the answer of both questions is immediately "yes", because graphs of polytopes (and hence graphs of its faces) are connected graphs.
>
> Moreover, one can ask for a *strong happy-edge property*: "Is the shortest path from C_1 to C_2 in Γ contained in $\Gamma|_{C_1 \cap C_2}$?", in a word, is keeping the shared part untouched the best way to flip from one configuration to another? If one benefits from a polytopal realization, then this amounts to ask for the shortest path between two vertices to be contained in the minimal face to which these two vertices belong. The later is called the *non-leaving-face property*.

⋆ Exercise 2.4.14 — [Volume of Z_G]
Show that the volume of a graphical zonotope is the number of its (labeled) spanning trees.

⋆ Exercise 2.4.15 — [A graphical zonotope with nice in-degree vector]
(With your computer, if you want.)

Let G be the graph on vertices $0, 1, 2, 3$ with edges 02, 03, 12, 13, 23.

Is Z_G a simple polytope ?

Compute the f-vector of Z_G. Deduce the h-vector of Z_G.

Find a direction $\boldsymbol{c}$ such that $\mathbf{in}_{\boldsymbol{c}}(\mathsf{Z}_G) = h_{\mathsf{Z}_G}$. Hint: $\boldsymbol{c} = (0, 0, 1, 2)$ is a nice direction.

‖ **(Semi-)open problem 2.4.16** *Characterize the graphs G such that there exists a direction $\boldsymbol{c}$ satisfying $\mathbf{in}_{\boldsymbol{c}}(\mathsf{Z}_G) = h_{\mathsf{Z}_G}$.*

APPENDICE 2.E

Hypergraphic polytopes, nestohedra

In the same flavor as for zonotopes, one can define Minkowski sums of (faces of) standard simplices. We will only define the counter-part of (weighted) graphical zonotopes, but the reader can easily imagine the rest. These polytopes are harder to study than zonotopes but still exhibit nice properties.

Definition 2.E.1 — [Hypergraphs]
An hypergraph $H = (V, C)$ is (just) a collection of subsets on n nodes, i.e. $C = (X_1, \dots, X_k)$ with $X_i \subseteq [n]$. The subsets X_i are called the blocks or hyperedges of the hypergraph.
Usually, one avoids $|X_i| = 0$ or $|X_i| = 1$ in an hypergraph.
When $|X_i| = 2$ for all i, the hypergraph is simply called a graph.
If for all $X, Y \in C$ with $X \cap Y \neq \varnothing$, one has $X \cup Y \in C$, then the hypergraph is called a building set.

Definition 2.E.2 — [Hypergraphic polytopes]
For $X \subseteq [n]$, we denote $\Delta_X := \mathrm{conv}\{\boldsymbol{e}_i \ ; \ i \in X\}$ with $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ the canonical basis of $\mathbb{R}^n$.
The hypergraphic polytope of the hypergraph $H = (V, C)$ is:

$$\mathsf{P}_H = \sum_{X \in C} \Delta_X$$

When H is a building set, then P_H is called a nestohedron.
The weighted version is defined naturally by endowing $H = (V, C)$ with a weight function $w : C \to \mathbb{R}_+$, and constructing $\mathsf{P}_{H,w} := \sum_{X \in C} w(X)\Delta_X$.

⋆ **Exercise 2.E.3**
Check that when H is a graph, then the (weighted) hypergraphic polytope is just the usual (weighted) graphical zonotope.
Show that all (weighted) hypergraphic polytopes are deformations of Π_n.

⋆ **Exercise 2.E.4**
Show that nestohedra are simple polytopes.

(Semi-)open problem 2.E.5 *(Probably not too complicated.) Characterize the simple hypergraphic polytopes.*

APPENDICE 2.F

Cayley polytopes, valuations: a proof of Euler's relation (without topology)

Definition 2.F.1 — [Valuations]
A weak valuation is a function ϕ which maps a polytope to some element of a group, such that for all hyperplanes H and polytope P (recall that H^+ and H^- are the two half-spaces defined by H):

$$\phi(\mathsf{P}) + \phi(\mathsf{P} \cap H) = \phi(\mathsf{P} \cap H^+) + \phi(\mathsf{P} \cap H^-)$$

A strong valuation is a function ψ which maps a polytope to some element of a group, such that for all polytopes P and Q, if $\mathsf{P} \cup \mathsf{Q}$ is a polytope, then:

$$\psi(\mathsf{P}) + \psi(\mathsf{Q}) = \psi(\mathsf{P} \cup \mathsf{Q}) + \psi(\mathsf{P} \cap \mathsf{Q})$$

Theorem 2.F.2 *If a function is a weak valuation on polytope, then it is a strong valuation on polytopes, and reciprocally. Hence, such a function is simply called a valuation.*

⋆ **Exercise 2.F.3**
Prove that, for polytopes, a function is a weak valuations if and only if it is a strong valuation.

⋆ **Exercise 2.F.4**
Show that the volume of a polytope, its number of integer points, the indicator function are valuation.

⋆ **Exercise 2.F.5**
Prove that Euler's characteristic $\chi(\mathsf{P}) = \sum_{i=0}^{d} (-1)^i f_i$ is a valuation.

Definition 2.F.6 — [Cayley polytopes]
For polytopes $\mathsf{P}_1, \dots, \mathsf{P}_k \subset \mathbb{R}^d$ their Cayley polytope is defined as the polytope in $\mathbb{R}^{k+d}$:

$$\mathsf{Cay}(\mathsf{P}_1, \mathsf{P}_2, \dots, \mathsf{P}_k) = \operatorname{conv}\Big(\{\boldsymbol{e}_1\} \times \mathsf{P}_1,\ \{\boldsymbol{e}_2\} \times \mathsf{P}_2,\ \dots,\ \{\boldsymbol{e}_k\} \times \mathsf{P}_k\Big)$$

Remark 2.F.7 Most often than not, this definition is applied for $k = 2$. It is then combinatorially equivalent to $\operatorname{conv}\Big(\{0\} \times \mathsf{P}_1, \{1\} \times \mathsf{P}_2\Big)$. For this reason, some authors like to define the general case via $\mathsf{Cay}(\mathsf{P}_0, \mathsf{P}_1, \dots, \mathsf{P}_k) = \operatorname{conv}\Big(\{\boldsymbol{0}\} \times \mathsf{P}_0, \{\boldsymbol{e}_1\} \times \mathsf{P}_1, \dots, \{\boldsymbol{e}_k\} \times \mathsf{P}_k\Big)$.

⋆ **Exercise 2.F.8**
Fix P, Q, and use the definition $\mathsf{Cay}(\mathsf{P}, \mathsf{Q}) = \operatorname{conv}\Big(\{0\} \times \mathsf{P}, \{1\} \times \mathsf{Q}\Big)$.

Show that the section of $\mathsf{Cay}(\mathsf{P}, \mathsf{Q})$ by an hyperplane $H_{\boldsymbol{e}_0,\lambda}$ for $\lambda \in]0,1[$ is normally equivalent to $\mathsf{P} + \mathsf{Q}$.

Deduce the f-vector of $\mathsf{Cay}(\mathsf{P}, \mathsf{Q})$ as a function of the f-vectors of P, Q and $\mathsf{P} + \mathsf{Q}$.

Deduce that $\chi(\mathsf{Cay}(\mathsf{P}, \mathsf{Q})) = \chi(\mathsf{P}) + \chi(\mathsf{Q}) - \chi(\mathsf{P} + \mathsf{Q})$.

Deduce that if $\chi(\mathsf{P}') = 1$ for all $(d-1)$-dimensional polytopes P', then $\chi(\mathsf{P}) = 1$ if P is a d-dimensional polytope that can be written as a Cayley polytope.

Fix a d-dimensional polytope P and a direction $\boldsymbol{c} \in \mathbb{R}^d$ such that no two vertices of P have the same scalar product with $\boldsymbol{c}$ (prove that such $\boldsymbol{c}$ exists). Cut P by the hyperplanes $H_{\boldsymbol{c},\langle \boldsymbol{v},\boldsymbol{c}\rangle}$ for $\boldsymbol{v} \in V(\mathsf{P})$. Conclude that $\chi(\mathsf{P}) = 1$ for all polytopes (*i.e.* prove Theorem 1.4.7), using the fact that χ is a valuation.

(Semi-)open problem 2.F.9 *Characterize which polytopes are (affinely equivalent) to Cayley polytope of length r, and how to efficiently recognize one.*

If you know lattice polytopes and h^*-vectors, note that tt has been proven that any lattice polytope whose h^*-polynomial is of degree s, is a Cayley polytope of lattice polytopes in dimension $\leq (s^2 + 19s - 4)/2$. Prove that the latter inequality can be replaced by $\leq s$. See Theorem 5.10 in Haase–Nill–Paffenholz's course.

CHAPTER

3

GRAPHS OF POLYTOPES

Content of the chapter

Section 3.1

Graphs, skeletons

Reminder from Section 1.3

Definition 3.1.1 — [(Directed) graph of a polytope]
The graph of a polytope $\mathsf{P} \subset \mathbb{R}^d$ *is the graph* G_{P} *whose node set is the vertex set* $V(\mathsf{P})$*, and whose arc set is the edge set* $E(\mathsf{P})$*.*

Once chosen a direction $\boldsymbol{c} \in \mathbb{R}^d$*, one can define the directed graph of a linear program* $(\mathsf{P}, \boldsymbol{c})$ *as the directed graph* $G_{\mathsf{P},\boldsymbol{c}}$ *whose node set is* $V(\mathsf{P})$ *and where there is an arc* $\boldsymbol{u} \to \boldsymbol{v}$ *if* $\boldsymbol{uv} \in E(\mathsf{P})$ *and* $\langle \boldsymbol{u}, \boldsymbol{c} \rangle < \langle \boldsymbol{v}, \boldsymbol{c} \rangle$*. The set of directed arcs of* $G_{\mathsf{P},\boldsymbol{c}}$ *is denoted* $E_{\boldsymbol{c}}(\mathsf{P})$*.*

A direction $\boldsymbol{c} \in \mathbb{R}^d$ *is called generic when* $\langle \boldsymbol{u}, \boldsymbol{c} \rangle \neq \langle \boldsymbol{v}, \boldsymbol{c} \rangle$ *for all edge* $\boldsymbol{uv} \in E(\mathsf{P})$*, that is to say when* $G_{\mathsf{P},\boldsymbol{c}}$ *is an orientation of* G_{P}*.*

⋆ **Exercise 3.1.2**
Among the following graphs, which ones would you guess are graphs of polytopes, and why? Complete graph, complete bipartite graph, path, cycle, star, wheel, graphs of (truncated) Platonic solids, Johnson graph, Petersen graph, Dürer graph, Herschel graph, Fullerene graph,...

Definition 3.1.3 — [Skeleton]
The k*-skeleton of a polytope* P *is the sub-poset of* $\mathscr{L}(\mathsf{P})$ *defined by all its* j*-faces for* $j \leq k$*.*

⋆ **Exercise 3.1.4**
For which polytopes is the k-skeleton of P a simplicial complex (for $k < \dim \mathsf{P}$)?

Definition 3.1.5 — [Incidence graphs]
The (i, j)*-incidence graph of a polytope* P *is the graph whose nodes set is the set of* i*-faces of* P*, i.e.* $\mathcal{F}_i(\mathsf{P})$*, and where two faces are linked if they belong to a common* j*-face.*

The k*-incidence graph of* P *is its* $(k, k+1)$*-incidence graph.*

The facet-incidence graph of P *is the line graph of its* $(d-2)$*-incidence graph, that is to say the graph whose node set is the set of facets of* P *and two facets are linked if they share a ridge (a co-dimension 2 face).*

Section 3.2

Diameter

Definition 3.2.1 — [Diameter]
The diameter $\delta(\mathsf{P})$ of a polytope P is the diameter of its graph, i.e. the maximum distance between two of its vertices.
We denote $\delta(d, n)$ to be the maximum diameter of d-polytopes with n vertices.

⋆ **Exercise 3.2.2**
Compute the diameter of simplices, cubes, hypersimplices, permutahedra, etc.

Theorem 3.2.3 *Here are some known bounds on n and d:*

Santos	*2011*	$\delta(d, n) > n - d$ for $d \geq 43$
Kalai & Kleitman	*1992*	$\delta(d, n) \leq n^{2+\log_2 d}$
Barnette 1967,	*Larman 1970*	$\delta(d, n) \leq \frac{1}{3}\, 2^{d-2} n$

Remark 3.2.4 To prove the first statement, Santos constructed a polytope (through *spindles*) with 86 facets in dimension 43. He then improved his example to (around) half this dimension, but the existence of a low-dimensional example is still open.

Conjecture 3.2.5 — [Polynomial Hirsch conjecture]
Is $\delta(d, n)$, the maximum diameter of d-polytopes with n vertices, bounded by a polynomial of n and d?

Section 3.3

Balinski's theorem

Definition 3.3.1 — [Connectedness of a graph]
A graph G is (vertex) d-connected if removing any $d-1$ of its nodes do not disconnect the graph.

Theorem 3.3.2 — [Menger, 1927]
If a graph is d-connected, then for any two vertices, there exists disjoint d paths linking them, where "disjoint" means no two paths share a node.

Remark 3.3.3 This theorem gives some background on connectivity, we do not prove it (we use it at the end of this chapter).

⋆ Exercise 3.3.4
Show that the nodes of a d-connected graph have degree at least d.

Show that the vertices of a d-polytope are of degree at least d in G_{P}.

Theorem 3.3.5 — [Balinski, 1961]
The graph of a d-polytope is d-connected. Indeed, if $X \subseteq V(\mathsf{P})$ disconnects G_{P}, then the affine dimension of X is $\geq d$.

⋆ Exercise 3.3.6
Show that the second sentence implies the first.

Fix a d-polytope P and a set $X \subseteq V(\mathsf{P})$ of affine dimension $\leq d-1$. Fix a vertex $\boldsymbol{u} \notin X$.

Show there is an hyperplane $H_{\boldsymbol{c},b}$ containing X and $\boldsymbol{u}$.

Let $V^+ = \{\boldsymbol{v} \in V(\mathsf{P})\ ;\ \boldsymbol{v} \notin X,\ \langle \boldsymbol{u}, \boldsymbol{c}\rangle \geq b\}$, and $V^- = \{\boldsymbol{v} \in V(\mathsf{P})\ ;\ \boldsymbol{v} \notin X,\ \langle \boldsymbol{u}, \boldsymbol{c}\rangle \leq b\}$. Finally, let $\boldsymbol{v}_{\min} \in V(\mathsf{P}^{-\boldsymbol{c}})$ and $\boldsymbol{v}_{\max} \in V(\mathsf{P}^{\boldsymbol{c}})$. Show that $V(\mathsf{P}) \smallsetminus X = V^+ \cup V^-$. Under which conditions $\boldsymbol{v}_{\min} \in X$ (resp. $\boldsymbol{v}_{\max} \in X$)? Show that any $\boldsymbol{v} \in V^+$ is connected to $\boldsymbol{v}_{\max}$ and reciprocally for V^-. Conclude.

⋆ Exercise 3.3.7
Find a d-polytope whose graph is not $(d+1)$-connected. *Hint: bi-pyramid*

Theorem 3.3.8 — [Perles & Prabhu, 1993]
For a d-polytope P and a k-face F, the graph $G_{\mathsf{P}} - V(\mathsf{F})$ is $\max(1,\ d-k-1)$-connected.

⋆ Exercise 3.3.9
Twist the proof of Balinski's theorem to prove Perles & Prabhu's theorem.

⋆ Exercise 3.3.10
Find a d-polytope and a k-face such that $G_{\mathsf{P}} - V(\mathsf{F})$ is not $(d-k-1)$-connected.

⋆ Exercise 3.3.11
Show that any k-face of a d-polytope has degree at least $(k+1)(d-k)$ in the k-incidence graph.

Theorem 3.3.12 — [Athanasiadis, 2009]
If $k \neq d-2$, then k-incidence graph of a d-polytope is $(k+1)(d-k)$-connected.

⋆ Exercise 3.3.13
Fix $k \geq 3$ (for the case $k=2$, look at Athanasiadis article [Ath09]). Let $\mathcal{G}_k(\mathsf{P})$ be the k-incidence graph of P. Pick $U \subseteq \mathcal{F}_k(\mathsf{P})$ with $\#U \leq (k+1)(d-k)-1$. For a face F of P, denote $U|_{\mathsf{F}} = \{\mathsf{G} \in U\ ;\ \mathsf{F} \subseteq \mathsf{G}\}$.

Consider the set $X := \Big\{\boldsymbol{v} \in V(\mathsf{P})\ ;\ \#U|_{\boldsymbol{v}} \geq k(d-k)\Big\}$. Fix $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{k+1} \in X$. Building the intersection one step at a time, show that $\#\bigcap_{i=1}^{k+1} U|_{\boldsymbol{v}_i} \geq k$, and deduce that the affine dimension of $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_{k+1})$ is at most $k-1$. Deduce that the affine dimension of X is at most $k-1 < d$.

Fix $\mathsf{F}, \mathsf{F}' \in \mathcal{F}_k \smallsetminus U$, and deduce from the previous that $V(\mathsf{F}) \smallsetminus X$ and $V(\mathsf{F}') \smallsetminus X$ are non-empty.

Hence, let $\boldsymbol{v}_{\mathsf{F}}, \boldsymbol{v}_{\mathsf{F}'} \notin X$ be vertices of F, F'. Show there exists a path $\mathcal{P}$ from $\boldsymbol{v}_{\mathsf{F}}$ to $\boldsymbol{v}_{\mathsf{F}'}$ in $G_{\mathsf{P}} - X$. *Balinski*

Using $k \geq 3$, show that $\binom{d-1}{k-1} \geq k(d-k)$. Deduce that each edge e of $\mathcal{P}$ is contained in $\leq \binom{d-1}{k-1} - 1$ elements of U. Deduce that each edge e of $\mathcal{P}$ is contained in a k-face $\mathsf{F}_{\mathsf{e}} \notin U$.

Consider two consecutive edges e, e' of $\mathcal{P}$, with $\boldsymbol{v}$ their common node. Remember that $\#U_{\boldsymbol{v}} < k(d-k)$, consider the $(k-1)$-incidence graph of the vertex figure $\mathsf{P}/\boldsymbol{v}$, and deduce that F_{e} and $\mathsf{F}_{\mathsf{e}'}$ are connected in $\mathcal{G}_k(\mathsf{P}) - U$.

Conclude by showing that $\mathcal{G}_k(\mathsf{P}) - U$ is connected.

(Semi-)open problem 3.3.14 — [Pineda-Villavicencio, 2024, [PV24, Problem 4.7.8]]
*Characterize d-polytopes whose graphs are critically d-connected (i.e. removing **any** d vertices disconnect the graph), and those whose graphs are minimally d-connected (i.e. there exists d vertices whose removable disconnect the graph).*

Section 3.4

Steinitz' theorem

◊ Construction 3.4.1 — [Schlegel diagram]

Consider a polytope P and facet F. Pick a point $\boldsymbol{p} \notin \mathsf{P}$ "close to F", that is to say $\boldsymbol{p}$ satisfies all the inequalities defining P except the one of F.

For every point $\boldsymbol{x} \in \mathsf{P}$, let $\boldsymbol{p} \cdot \boldsymbol{x}$ be the intersection of the segment $[\boldsymbol{p}, \boldsymbol{x}]$ with F. Show that $\boldsymbol{p} \cdot \boldsymbol{x}$ is well defined for all $\boldsymbol{x} \in \mathsf{P}$. Besides, let $\boldsymbol{p} \cdot \mathsf{G} = \{\boldsymbol{p} \cdot \boldsymbol{x} \ ; \ \boldsymbol{x} \in \mathsf{G}\}$ for a face $\mathsf{G} \neq \mathsf{F}$ of P. Show that $\boldsymbol{p} \cdot \mathsf{G}$ is a polytope if G is a polytope, and that it is combinatorially equivalent to G.

The *Schlegel diagram* of P on F (with respect to $\boldsymbol{p}$) is the subdivision of F defined by $\boldsymbol{p} \cdot \mathsf{G}$ for all faces $\mathsf{G} \neq \mathsf{F}$ of P. Show that the Schlegel diagram is indeed a polytopal subdivision (*i.e.* it is a collection of polytopes, all whose faces are also in the collection, and where two polytopes of the collection intersect on a common face).

The Schlegel diagram is naturally embedded in aff F, hence it is $(\dim \mathsf{P} - 1)$-dimensional, which is nice for making drawings. However, the drawback is that we get a subdivision: for a usual polytope, no-one care about its interior, but for a diagram, the interior is the most important part, and it is tricky to visualize.

Draw Schlegel diagrams of your favorite 3- and 4-dimensional polytopes.

⋆ Exercise 3.4.2

Thanks to the Schlegel diagram, show that the graph of a 3-polytope is a planar graph.

> **Theorem 3.4.3 — [(weak version of) Whitney, 1933, Tutte, 1963]**
> *If two 3-polytopes have the same graph, then they have the same face lattice.*

⋆ Exercise 3.4.4

A cycle in a graph is *non-separating* if its removal does not disconnect the graph.

Fix a 3-polytope P. Show that the graphs of its 2-faces are precisely the non-separating cycles of its graph.

Conclude.

⋆ Exercise 3.4.5

Show that a planar graph have a vertex of degree 5 or less. Deduce that every 3-polytopes have either a triangular, quadrangular, or pentagonal face.

> **Theorem 3.4.6 — [Steinitz, 1922]**
> *A graph is the graph of a 3-polytope if and only if it is planar and (vertex) 3-connected.*

Remark 3.4.7 In dimension 3, knowing if a graph $G = (V, E)$ is polytopal is easy: test its planarity and its 3-connectivity. This takes linear time in the product $|V| \cdot |E|$.

In higher dimensions ≥ 4, Universality theorem of Mnëv and Richter-Gebert [RG96, Theorem 9.1.2] implies that knowing if a graph is polytopal is **NP**-hard (it is a bit more subtle than that actually, and still unclear as far as I know: given a lattice, it is **NP**-hard to prove this lattice is polytopal, but given a graph, there may exists a lattice of faces inducing it whose polytopality is easy to decide).

⋆ Exercise 3.4.8

Show that the graph of a 3-polytope is planar and 3-connected.

We are not really going to prove the converse, and refer to [Zie95, Chapter 4].

Step 1: Prove that if G is a planar graph (you can suppose it has no vertex of degree 1 nor 2), then it has either a triangular face or a vertex of degree 3.

Step 2: If $\boldsymbol{v}$ if a vertex of degree 3 of a 3-polytope P, with neighbors $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$, then show that there exists another polytope Q, obtained by adding 1 inequality to P, such that $\boldsymbol{u}_1\boldsymbol{u}_2\boldsymbol{u}_3$ is a triangular face of Q and $\boldsymbol{v}$ has been removed from Q (be careful of the different cases).

Think about the dual/polar statement.

Think about its interpretation in the world of graphs (called a ΔY-move).

Step 3: *[that's the thing we are not going to do here !]* Prove that any planar 3-connected graph can be constructed from K_4 by performing a sequence of ΔY-moves, and conclude.

⋆ Exercise 3.4.9

Show that any graph is the sub-graph of the graph of a 4-dimensional polytope.

Take Cyc$(4, n)$ and its graph, then make some edges disappear by adding vertices.

Section 3.5

Reconstruction problems

Definition 3.5.1 — [Reconstruction problem]
We say that objects are reconstructible from their image under φ among a class $\mathcal{C}$ of objects if the application $\varphi : \mathcal{C} \to \mathcal{C}'$ is injective.

A reconstruction problem is hence a question like "Are 3-polytopes reconstructible from their graphs?", meaning "Does their exists two 3-polytopes which have the same graph but not the same face lattice?".

⋆ Exercise 3.5.2
Are 3-polytopes reconstructible from their graphs?

⋆ Exercise 3.5.3
Show that d-polytopes are reconstructible from their $(d-2)$-skeleton.

Theorem 3.5.4 — [Blind & Mani-Levitska 1987, Kalai 1988]
Simple polytopes are reconstructible from their graph.

⋆ Exercise 3.5.5 — [Kalai's proof]
For an acyclic orientation ρ of G_{P}, denote $L(\rho) = \sum_j h_j 2^j$ where h_j is the number of vertices of in-degree j in ρ.

Fix ρ. Double-count $(X, \boldsymbol{v})$ where $\boldsymbol{v} \in V(\mathsf{P})$ and X is a subset of in-going arcs at $\boldsymbol{v}$; especially, show that $L(\rho) \geq |\mathcal{F}(\mathsf{P})| - 1$ (*i.e.* the number of non-empty faces of P). What are the equality cases?

An acyclic orientation is *good* if $L(\rho) = |\mathcal{F}(\mathsf{P})| - 1$. Show that if $\boldsymbol{c}$ is generic, then $G_{\mathsf{P},\boldsymbol{c}}$ is a good orientation.

In an acyclic orientation, a sub-graph induced by $W \subseteq V$ is *initial* if all its xy with $x \in W$ and $y \notin W$ are oriented $x \to y$. Show that for all k-faces F of P, there exists a good orientation of G_{P} such that G_{F} is a k-regular initial sub-graph.

Finally, show that if an induced sub-graph H is a k-regular initial sub-graph in a good orientation ρ of G_{P}, then its set of vertices is the set of vertices of some k-face of P.

To rev

Hint: First look at the (unique) sink $\boldsymbol{v}$ of H, then take the incoming edges at $\boldsymbol{v}$ in H, and construct a k-face F out of them, and conclude that $G_{\mathsf{F}} = H$ by double-inclusion.

Conclude by proving the above theorem.

⋆ Exercise 3.5.6
How difficult is it to implement this proof in a computer in order to deduce a face lattice from a regular polytopal graph?

Theorem 3.5.7 — [Doolittle, Nevo, Pineda-Villavicencio, Ugon, Yost, 2019]
Polytopes with at most 2 non-simple vertices are reconstructible from their graph.

Remark 3.5.8 Read the paper to get the proof, [DNPV+19], it is an adaptation of Kalai's proof.

(Semi-)open problem 3.5.9 *Are polytopes with at most $d-2$ non-simple vertices reconstructible from their graph and dimension?*

⋆ Exercise 3.5.10 — [Joswig & Ziegler non-cube [JZ00, Section 4]]
Consider the following polytope $\mathsf{P} \subset \mathbb{R}^4$ (whose 32 vertices are in column):

−2	−2	−1	−2	−1	−2	−1	−1	1	1	2	1	1	1	2	2	−1	1	2	1	−1	2	1	2	−2	2	−2	2	−2	−1	−1	−2
−2	2	1	2	−1	−2	1	−1	1	1	−2	1	−1	1	2	−2	−1	−1	−2	−1	1	−2	−1	2	−2	2	−2	2	2	1	−1	2
−1	−1	−2	−1	−2	−1	−2	−2	2	2	1	−2	2	−2	−1	1	2	2	−1	−2	2	−1	−2	1	1	−1	1	1	1	2	2	1
−1	−1	2	1	2	1	−2	−2	2	−2	1	−2	−2	2	−1	−1	−2	2	−1	2	−2	1	−2	1	−1	1	1	−1	−1	2	2	1

Its inequalities are:

$$\begin{aligned} \pm x_i &\leq 2 && \text{for } i \in \{1,2,3,4\} \\ \pm x_i \pm x_j &\leq 3 && \text{for } i \in \{1,2\},\ j \in \{3,4\} \end{aligned}$$

Show (with your computer) that P is a 4-dimensional polytope whose graph is the graph of the 5-dimensional cube. Comment this fact in regard of the reconstructibility of simple polytopes.

Section 3.6

Catalan families, Associahedron

Perhaps the uttermost example of the interplay between combinatorics and polytopes is the link between Catalan families and the the associahedra. It gathers the quintessence of what we have studied so far: we will focus on combinatorics for some time before going back the polytopes.

Definition 3.6.1 — [Triangulations of n-gons]
A triangulation of a convex $(n+2)$-gon P_{n+2} is a set of n triangle whose vertices are among the vertices of P_{n+2} and which are not pairwise intersecting. The set of triangulations of P_{n+2} is denoted $\mathcal{T}_n$.

The diagonals of the triangulation are its $n-1$ interior edges (i.e. not the edges of P_{n+2}).

Two triangulations $T, T' \in \mathcal{T}_n$ differ by a flip if they share all but 1 diagonal. The graph of flips is the graph whose nodes are the triangulations, and arcs are between triangulations that differ by a flip.

⋆ **Exercise 3.6.2**
For a square, a pentagon and an hexagon, draw/count all triangulations, and draw the graph of flips.

Are these graphs of flips polytopal? If yes, comment on the combinatorics of the faces.

Definition 3.6.3 — [Catalan numbers]
The n-Catalan number is $Cat(n) = \frac{1}{n+1}\binom{2n}{n}$.

⋆ **Exercise 3.6.4 — [Catalan recursion and Catalan number]**
Label the vertices of P_{n+2} by $1, \dots, n+2$ in cyclic order.

Consider a triangulation and "cut it" using its triangle supported by the edge 1–$(n+2)$: show that the number $t_n = |\mathcal{T}_n|$ of triangulations of a $(n+2)$-gon satisfies $t_0 = 1$ and $t_n = \sum_{k=1}^{n} t_{k-1}\, t_{n-k}$. This recursive way of constructing triangulations is called the *Catalan recursion*.

Consider again the triangle supported by the edge 1–$(n+2)$, and contract it: show that $t_{n+1} = \sum_{T\in\mathcal{T}_n} \deg(T,1)$ where $\deg(T,1)$ is the number of neighbors of 1 in the triangulation T (excluding 2 and $n+2$).

Using rotational symmetry and the formula above, show $(n+2)t_{n+1} = 2(n+1)t_n$.

Conclude that $t_n = \mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$. Show that $\mathrm{Cat}(n) \sim \frac{4^n}{\sqrt{\pi}\, n^{3/2}}$ using Stirling's formula.

⋆ **Exercise 3.6.5 — [Catalan graph of flips]**
Show that the graph of flips of triangulations is connected: from a triangulation, go to a corner one.

Definition 3.6.6 — [Catalan families]
A Catalan family is a combinatorial family, i.e. for each n a finite set $\mathcal{C}_n$ of objects, such that $|\mathcal{C}_n| = Cat(n) = \frac{1}{n+1}\binom{2n}{n}$. Each element in a set $\mathcal{C}_n$ is called a Catalan object, and $\mathcal{C}_n$ is itself called a Catalan family of order n. Any Catalan family is hence in bijection with triangulations.

All Catalan families are endowed with a Catalan recursion, that is to say a recursive way to construct the objects in C_n which gives a proof the recursive formula $|\mathcal{C}_n| = \sum_{k=1}^{n} |\mathcal{C}_{k-1}|\,|\mathcal{C}_{n-k}|$.

A bijection between two Catalan families is a Catalan bijection when it factors though this recursion.

Each Catalan family is endowed with its own notion of flip, obtained as the image of the flip of triangulations through the Catalan bijection.

⋆ **Exercise 3.6.7 — [Some Catalan families]**
More than 200 Catalan families are presented in [Sta15], we discuss here some of them.

For each of the following Catalan families, draw the 2, 5 and 14 Catalan objects for $n = 1, 2, 3$.

- *Binary (search) trees*: trees on n nodes, where each node have either 0, 1 or 2 children, its left and right children. Nodes are labeled such that for every node, all its left descendants are smaller than itself, and all its right descendants are bigger. Equivalently, trees where each of the n internal nodes have 2 children (and leaves have none).
- *Parenthesizations*: ways to compute a (non-commutative) multiplication of $(n+1)$ factors.
- *Dyck paths, sub-staircase partition*: paths from $(0,0)$ to $(2n, n)$ which only uses steps $(1,+1)$ and $(1,-1)$; area under a broken line from $(0,0)$ to (n,n) using North steps $(0,1)$ and East steps $(1,0)$ which is never above the diagonal $(0,0)$–(n,n).
- *Dyck words, signed sequences with positive cumulants*: words of length $2n$ on the alphabet $\{+,-\}$ (or sometimes $\{X,Y\}$) such that, for all k, among the k first letters, the number of $+$ is always greater than the number of $-$.
- *Non-crossing partitions, nested pairings*: functions $A : [n-1] \to [n]$ with $A(i) > i$ and if $i < j < A(i)$ then $A(j) < A(i)$; partitions of $1, \dots, 2n$ into couples such that no two couples (i,j), (a,b) satisfy $i < a < j < b$.
- *Permutations avoiding* 312: permutations on n elements such that no three consecutive numbers $i\,j\,k$ with $j \le i \le k$.
- $2 \times n$ *standard Young tableaux*: ways of writing the number $1, \dots, 2n$ on two lines such that both lines are increasing, and all n columns are increasing.

Pick your favorite Catalan families, and look at the bijection between them.

Pick your favorite Catalan families, and understand the flip on it, *i.e.* the operation such that the flip graph on triangulation is isomorphic to the flip graph of your Catalan family through the bijection you have unraveled before.

Comment on the existence of other natural notions of "flip" appearing in Catalan families but different from the usual flip.

Comment on how the "rotational symmetry" that you see in triangulations embodies (or not) in other Catalan families.

Definition 3.6.8 — [Tamari lattice]
The flip can be directed to obtain a directed graph of flips: this graph is actually the Hasse diagram of a lattice, called the Tamari lattice. For a diagonal in a triangulation, look a the surrounding quadrilateral, says its corners are labeled with $i < j < k < \ell$, there are two possibilities: either the diagonal is i–j or it is j–ℓ. When flipping a triangulation, there is exactly 1 quadrangle which switches from one possibility to the other. A flip is positive if the former diagonal is j–ℓ and the new is i–k, and negative otherwise. The Tamari lattice is obtained by orienting the flip graph according to positivity.

⋆ Exercise 3.6.9

For the other Catalan families, describe the Tamari orientation of their flips.

Show that the Tamari lattice is a quotient of the weak order.

Definition 3.6.10
An n-associahedron, a.k.a. Stasheff n-polytope, is a polytope whose graph is the graph of flip of a Catalan families of order n.

Definition 3.6.11 — [Loday's associahedron]
Loday's associahedron is defined has the Minkowski sum $\mathsf{Asso}_n := \sum_{1\leq a<b\leq n} \Delta_{[a,b]}$ where $\Delta_X = \mathrm{conv}(\boldsymbol{e}_i \ ; \ i \in X)$.

Theorem 3.6.12 — [Loday, 2004]
Loday's associahedron is an associahedron.

⋆ Exercise 3.6.13

Show that Loday's associahedron is a deformed permutahedron, more precisely an hypergraphic polytope.

Show that Loday's associahedron is $\mathsf{Asso}_n = \left\{ \boldsymbol{x} \in \mathbb{R}^n \ ; \ \begin{array}{ll} \sum_{i=1}^n x_i = \binom{n+1}{2} & \\ \sum_{i\in I} x_i \geq \binom{|I|+1}{2} & \text{for all } \varnothing \subsetneq I = [a,b] \subsetneq [n] \end{array} \right\}$.

Remark 3.6.14 Recall that the standard permutahedron is $\Pi_n = \left\{ \boldsymbol{x} \in \mathbb{R}^n \ ; \ \begin{array}{ll} \sum_{i=1}^n x_i = \binom{n+1}{2} & \\ \sum_{i\in I} x_i \geq \binom{|I|+1}{2} & \text{for all } I \subsetneq [n] \end{array} \right\}$. Consequently, Loday's associahedron can be obtained by "removing" some inequalities from the description of the permutahedron: a polytope with such a property is called a *removahedron*.

⋆ Exercise 3.6.15

Show that Loday's associahedron has $\mathrm{Cat}(n)$ vertices, each one of them naturally corresponding to a Catalan object (binary trees are efficient to see that), and describe the normal cone at each vertex.

For a vertex $\boldsymbol{v}$ of Loday's associahedron, endowed with its binary tree T, show that the i-th coordinate is equal to $\ell(i) \cdot r(i)$ where $\ell(i)$ is the number of left descendants of the node i in T, and $r(i)$ its number of right descendants.

Show that the graph of Loday's associahedron G_{Asso_n} is isomorphic to the graph of flips of Catalan families.

Show that for $\boldsymbol{c} = (1, 2, \dots, n)$, the directed graph $G_{\mathsf{Asso}_n, \boldsymbol{c}}$ is (the Hasse diagram of) the Tamari lattice.

⋆ Exercise 3.6.16

From the previous exercise, deduce that Loday's associahedron is a simple polytope.

Deduce that Loday's associahedron is reconstructible from its graph, and that all associahedra are combinatorially equivalent.

This has an important consequence: there is a natural notion of faces on Catalan families! Describe this notion of faces for triangulations (called *subdivisions* of a polygon), binary trees (called *Schröder trees*), parenthesizations (called *partial parenthesizations*), non-crossing partitions, and permutations avoiding 312.

Prove that faces of the associahedron are products of associahedra.

⋆ Exercise 3.6.17

Prove that the graph of flips of Catalan families is n-connected.

Moreover, for any two Catalan objects C_1, C_2, let $C_1 \cap C_2$ be the common part of the objects (*e.g.* diagonals appearing in both triangulations C_1 and C_2, or sub-tree appearing in both binary trees C_1 and C_2, or parentheses appearing in both parenthesizations C_1 and C_2,...): show that there exists a flip-path linking C_1 and C_2 in which every Catalan object contains $C_1 \cap C_2$. This is a non-leaving face property! Show there are at least $n - |C_1 \cap C_2|$ disjoint such paths (use Menger's theorem).

(Semi-)open problem 3.6.18 — [Computing the distance between Catalan objects]
Design a polynomial-time algorithm which compute the distance (i.e. minimal number of flips necessary) between Catalan objects, or show there is no polynomial-time algorithm for this problem.

Remark 3.6.19 The diameter of the graph of flips of a Catalan family is known: $2n - 6$ for n large enough (but we do not know how “large” is sufficient).

CHAPTER

4

DEFORMATIONS OF POLYTOPES, MINKOWSKI INDECOMPOSABILITY

Content of the chapter

Section 4.1

Rehearsal

Definition 4.1.1 — [Deformation]
A deformation or (weak) Minkowski summand of P *is a polytope* Q *such that there exists* $\lambda > 0$ *and another polytope* R *satisfying:* $\mathsf{P} = \lambda\mathsf{Q} + \mathsf{R}$.

This is denoted $\mathsf{Q} \trianglelefteq \mathsf{P}$.

Definition 4.1.2 — [Minkowski indecomposable, Minkowski decomposition]
A polytope is said Minkowski indecomposable if all its deformations are dilates of itself.

A Minkowski decomposition of a polytope P *is a writing of* P *as a Minkowski sum of indecomposable polytopes. Two decompositions are the same if they use the same summands of* P.

⋆ **Exercise 4.1.3 — [Rehearsal]**
Prove that if $\mathsf{P}^{\mathbf{c}} = \mathsf{Q}^{\mathbf{c}} + \mathsf{R}^{\mathbf{c}}$, and hence if $\dim \mathsf{P}^{\mathbf{c}} = k$, then for any deformation Q of P, the face $\dim \mathsf{Q}^{\mathbf{c}} \leq k$.

Deduce that, if $\mathsf{Q} \trianglelefteq \mathsf{P}$, then for every edge e of Q, there exists (at least) an edge e' of P which is parallel to e. Show that this association is not necessarily injective.

⋆ **Exercise 4.1.4**
Prove that Q is a deformation of $\mathsf{P} \subset \mathbb{R}^d$ if and only if $\mathcal{N}_{\mathsf{Q}}$ coarsens $\mathcal{N}_{\mathsf{P}}$.

Deduce that if $\mathsf{Q} \trianglelefteq \mathsf{P}$, then Q has less vertices that P.

Prove that if Q, Q' are deformations of P, and $\lambda > 0$, then $\lambda\mathsf{Q} \trianglelefteq \mathsf{P}$, and $\mathsf{Q} + \mathsf{Q}' \trianglelefteq \mathsf{P}$. Deduce that all dilated translate of Q are deformations of P.

Deduce that the set of deformations of P is a cone (*i.e.* a set stable by sums and positive multiplication by a scalar).

Show that the *lineal* of this cone, *i.e.* the biggest vector space contained in this cone, is at least of dimension d.

⋆ **Exercise 4.1.5**
Show that simplices are Minkowski indecomposable.

Show that the only polygons which are indecomposable are triangles.

Definition 4.1.6 — [Deformation cone, Lattice of deformations]
The deformation cone $\mathbb{DC}(\mathsf{P})$ *of a polytope* P *is the set of all its deformations. The lattice of deformations of* P *is the poset whose elements are classes of normally equivalent deformations of* P*, ordered by* $\trianglelefteq$.

We will prove that the deformation cone is a (pointed) polyhedral cone, whose face lattice is the lattice of deformations.

Remark 4.1.7 To ease the notation, we will sometimes speak about the deformations, deformation cone, etc, of fans.

⋆ **Exercise 4.1.8**
Draw the deformation cone of a Minkowski indecomposable polytope.

Draw the deformation cone of the 2-dimensional permutahedron Π_3.

How many different (minimal) Minkowski decomposition of Π_3 exist?

Definition 4.1.9 — [Simplicial cones and fans]
A pointed cone is simplicial if it is a cone over a simplex. A fan is simplicial if all its cones are simplicial.

⋆ **Exercise 4.1.10**
If $\mathbb{DC}(\mathsf{P})$ is simplicial, then how many different Minkowski decompositions does P have?

Definition 4.1.11 — [Generalized permutahedra]
A generalized permutahedron, as known as deformed permutahedron, is a deformation of the standard permutahedron Π_n.

⋆ **Exercise 4.1.12**
What are the generalized permutahedra on dimension ≤ 3?

Section 4.2

Height vector

Definition 4.2.1 — [Height vector]
For a polytope P*, let* R *be the collection of rays of its normal fan, i.e. the (outer) normal vectors of the facets of* P*. The height vector* $\boldsymbol{h} \in \mathbb{R}^R$ *(sometimes called slack vector) of the polytope* P *is defined by its coordinates* $h_{\boldsymbol{r}} = \max_{\boldsymbol{x}\in\mathsf{P}} \langle \boldsymbol{x}, \boldsymbol{r}\rangle$*.*

⋆ Exercise 4.2.2
Show that the height vector of $\mathsf{P} + \mathsf{Q}$ is the (vector) sum of the height vector of P with the one of Q. What about $\lambda\mathsf{P}$?

Definition 4.2.3 — [$\mathsf{P}_{\mathcal{F},\boldsymbol{h}}$**,** $\mathsf{P}_{\boldsymbol{h}}$**]**
For a fan $\mathcal{F}$ *supported on ray set* R*, and* $\boldsymbol{h} \in \mathbb{R}^R$*, we denote* $\mathsf{P}_{\mathcal{F},\boldsymbol{h}} = \left\{\boldsymbol{x} \in \mathbb{R}^d \ ; \ \forall \boldsymbol{r} \in R,\ \langle \boldsymbol{x}, \boldsymbol{r}\rangle \leq h_{\boldsymbol{r}}\right\}$*. We usually denote* $\mathsf{P}_{\boldsymbol{h}}$ *for* $\mathsf{P}_{\mathcal{F},\boldsymbol{h}}$ *when* $\mathcal{F}$ *is clear from the context.*

⋆ Exercise 4.2.4
Show that the height vector gives the definition of P via inequalities, *i.e.* $\mathsf{P} = \mathsf{P}_{\mathcal{N}_\mathsf{P},\boldsymbol{h}}$ for $\boldsymbol{h}$ the height vector of P. Show that the inequalities in this description is irredundant.

⋆ Exercise 4.2.5
Suppose $\mathcal{F}$ is a complete simplicial fan. Let $\mathsf{C}_1, \mathsf{C}_2$ be two adjacent cones of $\mathcal{F}$, whose supporting ray sets are R_1 and R_2. Show that $|R_1| = |R_2| = d - 1$.
Let $S = R_1 \cap R_2$, show that $|S| = d - 1$. Let $\{\boldsymbol{r}_1\} = R_1 \smallsetminus S$ and $\{\boldsymbol{r}_2\} = S \smallsetminus R_2$.
Show that there exists $\boldsymbol{\alpha} \in \mathbb{R}^{(R_1\cup R_2)}$ with $\alpha_{\boldsymbol{r}_1} > 0$ and $\alpha_{\boldsymbol{r}_2} > 0$, and:

$$\sum_{\boldsymbol{r}\in S\cup\{\boldsymbol{r}_1,\boldsymbol{r}_2\}} \alpha_{\boldsymbol{r}}\boldsymbol{r} = \boldsymbol{0}$$

Definition 4.2.6 — [Wall-crossing inequalities]
For a complete simplicial fan $\mathcal{F}$ *supported on the ray set* R*, the wall-crossing inequality associated to two adjacent cones* $\mathsf{C}_1, \mathsf{C}_2$ *is the inequality on the coordinates of* $\boldsymbol{h} \in \mathbb{R}^R$ *given by:*

$$\sum_{\boldsymbol{r}\in S\cup\{\boldsymbol{r}_1,\boldsymbol{r}_2\}} \alpha_{\boldsymbol{r}} h_{\boldsymbol{r}} \geq 0$$

where C_i *is supported on ray set* $S \cup \{\boldsymbol{r}_i\}$ *for* $i \in \{1, 2\}$*.*

Theorem 4.2.7 — [Height parametrization of $\mathbb{DC}(\mathcal{F})$ **- simplicial version]**
For a simplicial fan $\mathcal{F}$ *on ray set* R*, its deformation cone* $\mathbb{DC}(\mathcal{F})$ *is the cone of* $\mathsf{P}_{\boldsymbol{h}}$ *for* $\boldsymbol{h} \in \mathbb{R}^R$ *satisfying all wall-crossing inequalities of* $\mathcal{F}$*.*

⋆ Exercise 4.2.8
Convince yourself that this is the correct formula in 2 dimensions and in 3 dimensions. For the formal proof, go see the work of Fomin–Chapoton–Zelevinsky, or work it out yourself.
Note that the main idea is asking yourself which formula governs the (regular) triangulation of the union of two adjacent simplicial cones.

Theorem 4.2.9 *For any fan* $\mathcal{F}$*, there exists a simplicial fan* $\mathcal{G}$ *who refines* $\mathcal{F}$ *(using the same set of rays).*

⋆ Exercise 4.2.10
Wait for Chapter 5 to learn that any polytope can be triangulated. Deduce this theorem.

Theorem 4.2.11 — [Height parametrization of $\mathbb{DC}(\mathsf{P})$**]**
Let $\mathsf{P} \subset \mathbb{R}^d$ *be a polytope (not necessarily simple), and* $\mathcal{G}$ *a simplicial fan refining* $\mathcal{N}_\mathsf{P}$*, supported on ray set* R*. Then its deformation cone* $\mathbb{DC}(\mathsf{P})$ *is the cone of* $\mathsf{P}_{\boldsymbol{h}}$ *for* $\boldsymbol{h} \in \mathbb{R}^R$ *satisfying both of the following:*

- *the wall-crossing equalities* $\sum_{\boldsymbol{r}\in S\cup\{\boldsymbol{r}_1,\boldsymbol{r}_2\}} \alpha_{\boldsymbol{r}} h_{\boldsymbol{r}} = 0$ *for adjacent cones* $\mathsf{C}_1, \mathsf{C}_2$ *of* $\mathcal{G}$ *belonging to* ***the same*** *cone of* $\mathcal{F}$*;*
- *the wall-crossing inequalities* $\sum_{\boldsymbol{r}\in S\cup\{\boldsymbol{r}_1,\boldsymbol{r}_2\}} \alpha_{\boldsymbol{r}} h_{\boldsymbol{r}} \geq 0$ *for adjacent cones* $\mathsf{C}_1, \mathsf{C}_2$ *of* $\mathcal{G}$ *belonging to* ***different*** *cone of* $\mathcal{F}$*.*

Corollary 4.2.12 *The deformation cone is a polyhedral cone, and the set of deformations is a lattice for the order* $\trianglelefteq$*.*

⋆ Exercise 4.2.13
Prove Theorem 4.2.11 and Corollary 4.2.12.

Theorem 4.2.14 — [Faces of $\mathbb{DC}(\mathsf{P})$]
For a polytope P, the set of polytopes which are normally equivalent to P is in bijection with the interior of $\mathbb{DC}(\mathsf{P})$.
If $\mathsf{Q} \trianglelefteq \mathsf{P}$, then $\mathbb{DC}(\mathsf{Q})$ is a face of $\mathbb{DC}(\mathsf{P})$: this face is precisely the face of $\mathbb{DC}(\mathsf{P})$ in the interior of which Q lies.

Remark 4.2.15 This theorem allows to "easily" compute faces of $\mathbb{DC}(\mathsf{P})$: suppose there exists $\mathsf{Q} \trianglelefteq \mathsf{P}$ which is a simple polytope (either "simple" in the mathematical sense, or in the usual English sense), then $\mathbb{DC}(\mathsf{Q})$ is usually easier to compute than $\mathbb{DC}(\mathsf{P})$ itself, and is a face of it.

⋆ Exercise 4.2.16
Using Theorem 4.2.11 and the coarsening of normal fans, prove this theorem.

Theorem 4.2.17 — [Dimension of deformation cones]
For a polytope P with f_{d-1} facets, one has $\dim \mathbb{DC}(\mathsf{P}) \leq f_{d-1}$. The equality is achieved exactly for simple polytopes.

⋆ Exercise 4.2.18
Prove this theorem.

⋆ Exercise 4.2.19
Design an algorithm to test whether a fan is polytopal or not. Discuss the complexity of this algorithm.

Remark 4.2.20 For a fan $\mathcal{F}$, checking if $\mathcal{F}$ is polytopal can be solved in polynomial time in the number of maximal cones of $\mathcal{F}$ and the dimension it lives in. However, computing the dimension of $\mathbb{DC}(\mathcal{F})$ remains hard in practice. Moreover, most "interesting" fans have an exponential (or more) number of maximal cones.

⋆ Exercise 4.2.21
Show that the normal vectors of a generalized permutahedron $\mathsf{P} \subset \mathbb{R}^n$ are $-\boldsymbol{e}_A$ for some $A \subseteq [n]$, *i.e.* there exists $\boldsymbol{h} \in \mathbb{R}^{2^{[n]} \smallsetminus \{\varnothing, [n]\}}$ with $\mathsf{P} = \{\boldsymbol{x} \in \mathbb{R}^n ~;~ \sum_{i \in A} x_i \geq h_A\}$. Deduce that the cube and the associahedron are generalized permutahedra.

Definition 4.2.22 — [Submodular functions, Submodular cone]
A set function $f : 2^{[n]} \to \mathbb{R}$ is said to be submodular if it satisfies:

$$\forall A, B \subseteq [n],~ f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$$

The submodular cone $\mathbb{SC}_n$ is the set of submodular functions on $[n]$.

⋆ Exercise 4.2.23
Show that f is submodular iff it respects the sub-additive formula, iff it respects the diminishing returns property:

sub-additive formula:	$\forall X \subset [n]$,	$\forall x, x' \in [n] \smallsetminus X$,	$f(X \cup \{x\}) + f(X \cup \{x'\})$	$\geq$	$f(X) + f(X \cup \{x, x'\})$
diminishing return property:	$\forall X \subseteq Y \subseteq [n]$,	$\forall x \in [n] \smallsetminus Y$,	$f(X \cup \{x\}) - f(X)$	$\geq$	$f(Y \cup \{x\}) + f(Y)$

⋆ Exercise 4.2.24
Show that submodular functions form is a cone (using the operations $\lambda f : X \mapsto \lambda f(X)$ and $f + g : X \mapsto f(X) + g(X)$).
Show that the cone of generalized permutahedra is (linearly isomorphic to) the submodular cone, *i.e.* $\mathbb{DC}(\Pi_n) = \mathbb{SC}_n$.

⋆ Exercise 4.2.25
Give a facet description of $\mathbb{DC}(\Pi_3)$, and show (again) that it is a bi-pyramid over a triangle. Interpret every face as the deformation cone of some generalized permutahedra.

⋆ Exercise 4.2.26
What is $\dim \mathbb{SC}_n$? Its number of facets? Is it simplicial? Give (stupid) lower and upper bounds on its number of rays.

Remark 4.2.27 Some faces of the submodular cone are known, but its number of rays is widely open since Edmonds' article in 1970. We know that the number of rays t_n satisfies is: $\log\log t_n \geq n - \frac{3}{2}\log n + O(1)$ (yes, there are two logs). Recent results from Loho, Padrol & I show: $\log\log t_n \geq n$, and the real asymptotic is probably far greater. Simplices, shard polytopes, and more generally matroid polytopes for connected matroids are among the rays.
Besides, a facet description of some faces of $\mathbb{SC}_n$ is known, namely the one associated to Loday's associahedron, to graphical zonotopes, to nestohedra, and some other sparse examples (and quotientopes are almost done...). Note that $\mathbb{DC}(\mathsf{Asso}_n)$ is simplicial, and $\mathbb{DC}(\mathsf{Z}_G)$ is simplicial for a graph G without triangle.
Rays of the known simplicial faces are known, and rays of $\mathbb{DC}(\mathsf{Z}_G)$ for a graph G without K_4 are known.

⋆ Exercise 4.2.28
Show that the cone of (weighted) graphical zonotopes is non a deformation cone, but is the intersection of $\mathbb{SC}_n$ by a vector space of dimension $\binom{n}{2}$. What about the cone of (weighted) hypergraphic polytopes?

(Semi-)open problem 4.2.29 *Compute explicitly $\mathbb{SC}_n$ for $n \geq 5$ (and in other types).*

Section 4.3

Edge-length vector

Definition 4.3.1 — [Edge-length vector]
For a polytope P*, its edge-length vector is* $\boldsymbol{\ell} \in \mathbb{R}_+^{E(\mathsf{P})}$ *where* $\boldsymbol{\ell}_{\mathsf{e}}$ *is the length of the edge* $\mathsf{e} \in E(\mathsf{P})$*, i.e.* $\ell_{[\boldsymbol{u},\boldsymbol{v}]} = \|\boldsymbol{u} - \boldsymbol{v}\|$.

Definition 4.3.2 — [Edge-deformation vector]
For a deformation Q *of* $\mathsf{P} \subseteq \mathbb{R}^d$*, recall that for each vertex* $\boldsymbol{p} = \mathsf{P}^{\boldsymbol{c}}$ *of* P*, there exists a vertex* $\boldsymbol{q} = \mathsf{Q}^{\boldsymbol{c}}$ *of* Q *which is naturally associated to* $\boldsymbol{p}$*. Hence, it make sense to define the edge-length deformation vector to be* $\boldsymbol{\lambda} \in \mathbb{R}_+^{E(\mathsf{P})}$ *by its coordinate, for* $\boldsymbol{p}, \boldsymbol{p}'$ *two adjacent vertices of* P*, let* $\boldsymbol{q}, \boldsymbol{q}'$ *be the corresponding vertices of* Q*, then:*

$$\lambda_{\boldsymbol{p},\boldsymbol{p}'} = \frac{\|\boldsymbol{q}' - \boldsymbol{q}\|}{\|\boldsymbol{p}' - \boldsymbol{p}\|}$$

Note that, in the literature, only the name "edge-length vector" exist: we give here two separate definitions to make the picture clear for the reader, but we will now forget the name "edge-deformation vector" are use only "edge-length vector" for both definitions.

⋆ **Exercise 4.3.3**
Make sure that you have understood the definition and that you are sure it is well defined.

⋆ **Exercise 4.3.4**
Show that the edge-length vector of $\mathsf{P}+\mathsf{Q}$ is the (vector) sum of the edge-length vector of P with the one of Q. How about $\lambda\mathsf{P}$?

⋆ **Exercise 4.3.5**
Show that P is indecomposable if and only if the set of edge-length vectors of its deformations is the ray in direction $(1, \dots, 1)$.

Definition 4.3.6 — [$\mathsf{P}_{\boldsymbol{p},\boldsymbol{\lambda}}$, $\mathsf{P}_{\boldsymbol{\lambda}}$]
For a polytope P*, a fixed vertex* $\boldsymbol{p} \in V(\mathsf{P})$ *and* $\boldsymbol{\lambda} \in \mathbb{R}_+^{E(\mathsf{P})}$*, we define* $\mathsf{P}_{\boldsymbol{p},\boldsymbol{\lambda}}$ *by:*

$$\mathsf{P}_{\boldsymbol{p},\boldsymbol{\lambda}} = \operatorname{conv}\left\{ \sum_{\boldsymbol{p}_i\boldsymbol{p}_j \in (\boldsymbol{p} \rightsquigarrow \boldsymbol{p}')} \lambda_{\boldsymbol{p}_i,\boldsymbol{p}_j} (\boldsymbol{p}_j - \boldsymbol{p}_i) \;\; ; \;\; \boldsymbol{p}' \in V(\mathsf{P}) \right\}$$

where $(\boldsymbol{p} \rightsquigarrow \boldsymbol{p}') = (\boldsymbol{p} = \boldsymbol{p}_0, \boldsymbol{p}_1, \dots, \boldsymbol{p}_r = \boldsymbol{p}')$ *is a path from* $\boldsymbol{p}$ *to* $\boldsymbol{p}'$ *in the graph of* P *(which is connected).*
When $\boldsymbol{p}$ *is clear from the context or irrelevant, we simply write* $\mathsf{P}_{\boldsymbol{\lambda}}$.

⋆ **Exercise 4.3.7**
Check that the definition of $\mathsf{P}_{\boldsymbol{p},\boldsymbol{\ell}}$ does not depend on the choice of the path $(\boldsymbol{p} \rightsquigarrow \boldsymbol{p}')$.

⋆ **Exercise 4.3.8**
Show that changing the starting vertex $\boldsymbol{p}$ in $\mathsf{P}_{\boldsymbol{p},\boldsymbol{\ell}}$ amounts to translating the resulting polytope.

⋆ **Exercise 4.3.9**
Let P be a polygon, and $\boldsymbol{p}_1, \dots, \boldsymbol{p}_r$ an enumeration of its vertices in cyclic order. Show that $\sum_{i=1}^r \boldsymbol{p}_i - \boldsymbol{p}_{i+1} = \boldsymbol{0}$ (with the convention $\boldsymbol{p}_{r+1} = \boldsymbol{p}_1$).
Show that the edge-length vector of any deformation Q of P satisfy $\sum_{i=1}^r \lambda_{\boldsymbol{p}_i,\boldsymbol{p}_{i+1}} (\boldsymbol{p}_i - \boldsymbol{p}_{i+1}) = \boldsymbol{0}$ (with the convention $\boldsymbol{p}_{r+1} = \boldsymbol{p}_1$).

Definition 4.3.10 — [Polygonal-face equation]
For a polytope P *and a 2-face* F *with edges in cyclic order* $\mathsf{e}_1 = \boldsymbol{p}_1\boldsymbol{p}_2$*, ...,* $\mathsf{e}_{r-1} = \boldsymbol{p}_{r-1}\boldsymbol{p}_r$*,* $\mathsf{e}_r = \boldsymbol{p}_r\boldsymbol{p}_1$*, the associated polygonal-face equation is the equation on the coordinates of* $\boldsymbol{\lambda} \in \mathbb{R}^{E(\mathsf{P})}$ *given by:*

$$\sum_{i=1}^{r} \lambda_{\boldsymbol{p}_i,\boldsymbol{p}_{i+1}} (\boldsymbol{p}_i - \boldsymbol{p}_{i+1}) = \boldsymbol{0}$$

(with the convention $\boldsymbol{p}_{r+1} = \boldsymbol{p}_1$*)*

Theorem 4.3.11 — [Edge-length parametrization of $\mathbb{DC}(\mathsf{P})$]
Let $\mathsf{P} \subset \mathbb{R}^d$ *be a polytope, then its deformation cone* $\mathbb{DC}(\mathsf{P})$ *is the cone of (all the translate of)* $\mathsf{P}_{\boldsymbol{\lambda}}$ *for* $\lambda \in \mathbb{R}^{E(\mathsf{P})}$ *satisfying all the polygonal-face equations of* P*, and* $\lambda_{\boldsymbol{pp}'} \geq 0$ *for all edge* $\boldsymbol{pp}'$ *of* P.

⋆ **Exercise 4.3.12**
Prove this theorem.

⋆ **Exercise 4.3.13**
Comment the differences (assets and drawbacks) between the height parametrization, the edge-length parametrization and the polytope definition of the deformation cone: write clearly the 3 linearly isomorphic versions of this cone.

Remark 4.3.14 There are (at least) two other versions of this cone: one using (ample) divisors of the toric variety of P, and the other one as $\mathbb{DC}(\mathsf{P}) \simeq \bigcap_{S \text{ co-facet of } \mathcal{A}} \operatorname{conv}\{\boldsymbol{b}\ ;\ \boldsymbol{b} \in S\}$ where $\mathcal{A}$ is the vertex set of P° and $\operatorname{Gale}(\mathcal{A}) = \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_r\}$ is the Gale transform of $\mathcal{A}$.

Theorem 4.3.15 — [Dimension of deformation cones]
For a polytope P with f_1 edges, one has $\dim \mathbb{DC}(\mathsf{P}) \leq f_1$.

⋆ **Exercise 4.3.16**
Prove this theorem. Comment in regards of Theorem 4.2.17

⋆ **Exercise 4.3.17**
Design and algorithm to decide if a fan is polytopal or not.

⋆ **Exercise 4.3.18 — [Characterization of generalized permutahedra]**
Show that $\mathsf{P} \subset \mathbb{R}^n$ is a generalized permutahedron if and only if its edges are in direction $\boldsymbol{e}_i - \boldsymbol{e}_j$ for some $i, j \in [n]$.

Show that this property is indeed satisfied by the permutahedron, by graphical zonotopes, by Loday's associahedron, ...

Remark 4.3.19 This is the shortest way to define generalized permutahedra, so you will probably encounter it in talks.

⋆ **Exercise 4.3.20**
For the last time, compute $\mathbb{DC}(\Pi_3)$, using the edge-length parametrization.

⋆ **Exercise 4.3.21 — [Deformation cone of parallelotopes]**
Write down the polygonal face equation of a parallelogram.

Let $\mathsf{P} \subset \mathbb{R}^d$ be a parallelotope (*i.e.* a zonotope generated by d linearly independent vectors), and Q a deformation of P. Show that knowing the length of d edges of Q, one can deduce the length of all its edges.

Deduce that $\mathbb{DC}(\mathsf{P})$ is a simplicial cone of dimension d.

Deduce that there is a unique way to write a parallelotope as a sum of indecomposable polytopes (and this is via its definition as a zonotope).

Remark 4.3.22 This idea goes actually far beyond this exercise: on the one side, the deformation cone of any polytope which is combinatorially equivalent to a product of simplices, is a simplicial cone [CDG+22] ; on the other side, the deformation cone of any zonotope whose 2-faces are parallelograms, is simplicial [Padrol & I, 25+].

(Semi-)open problem 4.3.23 — [Simplicial deformation cone for a polytope with large polygonal faces]
For each dimension d, construct a d-polytope P whose deformation cone $\mathbb{DC}(\mathsf{P})$ is simplicial, but whose all 2-faces are neither triangles nor quadrangles. For all $k \geq 5$, is it possible to have $\mathbb{DC}(\mathsf{P})$ simplicial for P whose 2-faces are polygons with more than k vertices?

Section 4.4

Minkowski indecomposability

⋆ **Exercise 4.4.1**
Show that any polytope can be written as a Minkowski sum of Minkowski indecomposable polytopes. How many ways are there to do so?

♢ **Construction 4.4.2 — [Indecomposability game]**
To know if a polytope P is indecomposable or not, one should play the following *indecomposability game*. Fix a vertex $\boldsymbol{p}$ of P (in the sense that you assume to know exactly its coordinates), and the length of one edge of P. You are allowed to take whatever vertex and edge length that are the most suitable for you. From this knowledge, try to deduce the coordinates of other vertices (and the length of other edges) of P.

If you manage to deduce the coordinate of all the vertices, or equivalently the length of all the edges, then your polytope is indecomposable. If not, you at least will grasp an upper bound on the dimension of $\mathbb{DC}(\mathsf{P})$ (be careful, this upper bound is not as straightforward as it may seem).

Theorem 4.4.3 — [Indecomposability of simplicial polytopes]
All simplicial polytopes are Minkowski indecomposable.

⋆ **Exercise 4.4.4**
Show that simplices are indecomposable, using the polygonal face equation of triangles, and the indecomposability game.

Deduce that all simplicial polytopes are indecomposable, using the indecomposability game. Deduce that the convex hull of randomly chosen points is almost surely indecomposable.

Comment this theorem in regard of the case of simple polytopes, cf Theorem 4.2.17.

Moreover, show that if all the 2-faces of P are triangles, then P is indecomposable.

⋆ **Exercise 4.4.5**
Write the polygonal face equation of a parallelogram.

Consider the standard cube, remove one vertex (*i.e.* consider the convex hull of all but one vertex of the cube), to obtain the *Strawberry*.

Prove that the Strawberry is indecomposable.

Consider the graphical zonotope of the 4-cycle. Remove one vertex of degree 4 to obtain the *Persimmon* ; remove the two antipodal degree-4 vertices to obtain the *cuboctahedron*.

Show that the Persimmon is indecomposable, whereas the cuboctahedron is not (how to write is as a Minkowski sum?).

Theorem 4.4.6 *If* P *has a non-empty indecomposable face which share a vertex with every facets, then* P *is indecomposable.*

⋆ **Exercise 4.4.7**
Let's play the indecomposability game.

Let F be the indecomposable face at stake. Start with $\boldsymbol{p} \in V(\mathsf{F})$, and (if $\dim \mathsf{F} \geq 1$) with the length of one edge of F.

Deduce the coordinate of all the vertices of F (even if $\dim \mathsf{F} = 0$).

Deduce the height function of all the facets of P.

Deduce that P is indecomposable.

Corollary 4.4.8 — [Shepard, 1963]
If there exists a family of indecomposable faces $\mathcal{F} = (\mathsf{F}_1, \dots, \mathsf{F}_k)$ such that every facet share a vertex with one of the F_i, and such that the following graph $G_\mathcal{F}$ is connected, then P *is indecomposable. The nodes of $G_\mathcal{F}$ are the F_i, and F_i is linked with F_j by an arc in $G_\mathcal{F}$ if $\dim(\mathsf{F}_i \cap \mathsf{F}_j) \geq 1$.*

⋆ **Exercise 4.4.9**
Deduce this Corollary from Theorem 4.4.6.

Definition 4.4.10 — [Shard polytopes]
For $A \subset [n]$ with $1 \in A$ and $n \notin A$, an A-alternating matching is an increasing sequence sequence $M = (a_1 < b_1 < a_2 < b_2 < \dots, a_r < b_r)$ with $a_i \in A$, $b_j \in [n] \smallsetminus A$. The shard polytope $\mathsf{SP}(A)$ associated to A is the convex hull of $\boldsymbol{e}_M = \sum_{a \in M \cap A} \boldsymbol{e}_a - \sum_{b \in M \smallsetminus A} \boldsymbol{e}_b$ for M an A-alternating matching.

⋆ **Exercise 4.4.11**
Prove that, up to translation, a shard polytope is a 0/1-polytope.

Prove that a shard polytope is a generalized permutahedron.

An *A-fall* is $j \in A$ such that $j+1 \notin A$. An *A-rise* is $j \notin A$ or $j = 1$ such that $j+1 \in A$ or $j+1 = n$.

Show that the following is the facet description of the shard polytope $\mathsf{SP}(A)$:

$$\mathsf{SP}(A) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;;\; \begin{array}{ll} \sum_{i=1}^{n} x_i = 0 & \\ x_a \geq 0 & \text{for } a \in A \\ x_b \leq 0 & \text{for } b \notin A \\ \sum_{i \leq f} x_i \leq 1 & \text{for } f \text{ an } A\text{-fall} \\ \sum_{i \leq r} x_i \geq 0 & \text{for } r \text{ an } A\text{-rise} \end{array} \right\}$$

Prove that there is an edge between $\boldsymbol{e}_\varnothing$ and $\boldsymbol{e}_{(1,n)}$, and that this edge shares a vertex with every facet of $\mathsf{SP}(A)$.
Deduce that $\mathsf{SP}(A)$ is indecomposable.
Deduce that the cone $\mathbb{SC}_n$ has at least $2^n - n - 1$ rays. Comment that this lower bound was obvious...

Remark 4.4.12 What is not obvious is the following: shard polytopes form a basis of the ambient vector space in which $\mathbb{SC}_n$ lies. The same is true for the family $\big(\Delta_X \;;\; \varnothing \neq X \subset [n],\ |X| \geq 2\big)$.

Remark 4.4.13 Shard polytopes are fundamental object in the study of the lattice properties of the weak order (and its siblings): for any quotient of the weak order, its Hasse diagram is the graph of a generalized permutahedron, which can be realized as the sum of well-chosen shard polytopes. Besides, similar constructions can be used to give a polytopal realization to plenty of lattices (for other types, for polytopal complexes, ...). We could have made an exercise on quotientopes, but it would be too long to be included here (perhaps another time).

(Semi-)open problem 4.4.14 — [Smilansky, 1986]
For dimension 3, both of the following are impossible, but what about other dimensions:
Is it possible to create an indecomposable polytope which has no triangles?
It is possible to create an indecomposable polytope whose facets are cubes?

APPENDICE 4.E

Some data on the submodular cones

With the help of a computer, one can construct $\mathbb{SC}_4$ and partially $\mathbb{SC}_5$, we include here some information, just to give a sense of how big the problem is. Note that this section is meant to be out-dated: it has been written in late 2024, and already some better results can be included (coming from generalized polymatroids for instance).

***f*-vectors** The cone $\mathbb{SC}_4$ has 80 facets indimension 11. After (a lot of) computation, one can obtain that $\mathbb{SC}_4$ has 22 107 faces, each one of them corresponding to a different (class of normally equivalent) generalized permutahedron. However, some of these generalized permutahedra are equivalent up to central symmetry or permutations of coordinates: once this identification done, it "only" remains 703 different generalized permutahedra which are deformations of the 3-dimensional Π_4.

The reduced f-vectors of $\mathbb{SC}_n$ are ($n = 5$ was computed with Winfried Bruns):
$\mathbb{SC}_3$: (2, 2, 1, 1)
$\mathbb{SC}_4$: (7, 25, 64, 127, 174, 155, 97, 39, 12, 2, 1)
$\mathbb{SC}_5$: 672
24 026
373 433
3 355 348
19 739 627
81 728 494
249 483 675
579 755 845
1 048 953 035
1 501 555 944
1 719 688 853
1 587 510 812
1 186 372 740
719 012 097
353 190 577
140 265 886
44 831 594
11 464 559
2 326 596
372 031
46 330
4 572
355
30
2
1

Rays You already know the 2 rays (up to reduction) of $\mathbb{SC}_3$ correspond to a segment and a triangle.

Show that the 7 rays of $\mathbb{SC}_4$ correspond to: a segment, a triangle, a tetrahedron, an octahedron, a pyramid over a square, the Strawberry (convex hull of all but one vertices of a cube), the Persimmon (convex hull of all but a degree-4 vertex of the graphical zonotope of the 4-cycle).

All matroid polytopes for a connected matroid correspond to rays of $\mathbb{SC}_n$, but numerical experiments seem to show that, asymptotically, this type of rays amounts for a negligible part of the total of the rays. Loho, Padrol I [25⁺] have proven this statement by constructing more than c^{2^n} for some $c > 2$. Note that matroid polytopes correspond to 0/1 generalized permutahedra.

> **Conjecture 4.E.1 — [Combinatorially equivalent indecomposable generalized permutahedra]**
> *If* P, Q *are two indecomposable generalized permutahedra which are combinatorially equivalent, then* Q *can be obtained from* P *by permuting coordinates and taking central symmetry (if necessary).*
> *Note that this is false for decomposable generalized permutahedra.*

Equilateral polytopes A polytope is *equilateral* if all its edges have the same length.

This is in particular interesting for indecomposable polytopes: if P is equilateral and indecomposable, then all its deformations are dilates of P, hence also equilateral.

Note that all the rays of $\mathbb{SC}_4$ correspond to equilateral polytopes! This is no longer true for $\mathbb{SC}_5$, but there seem to be a surprisingly large number of them which are equilateral.

> **(Semi-)open problem 4.E.2 — [Equilateral indecomposable generalized permutahedra]**
> *Characterize (or at least construct a lot of) equilateral indecomposable generalized permutahedra.*

Special faces of $\mathbb{SC}_n$***, and simplicial faces*** Fixing a generalized permutahedron P and computing its deformation cone is equivalent to computing a face of $\mathbb{SC}_n$. This has been done in the case of Loday's associahedra, graphical zonotopes, nestohedra, and is almost done for quotientopes.

Especially, in theses cases we know which graph (respectively building set and arc diagram) give rise to a simplicial face of $\mathbb{SC}_n$, hence which of these generalized permutahedra have a unique way to be written as a Minkowski sum.

(Semi-)open problem 4.E.3 *Find a large class of simplicial faces of* $\mathbb{SC}_n$.

(Semi-)open problem 4.E.4 — [Deformation cone of hypergraphic polytopes]
For an hypergraph H*, compute the deformation cone of its hypergraphic polytope* $\mathbb{DC}(\mathsf{P}_H)$*. Determine for which hypergraph this cone is simplicial.*

Cubal relations For $S \subseteq [n]$ and $x \in S$, we denote $Sx = S \smallsetminus \{x\}$.

The normal vector to the facets of $\mathbb{SC}_n$ are $\boldsymbol{n}(S,u,v) = \boldsymbol{e}_{Suv} - \boldsymbol{e}_{Su} - \boldsymbol{e}_{Sv} + \boldsymbol{e}_S$, for every $u, v \in S \subseteq [n]$, where $(\boldsymbol{e}_X \ ; \ X \subseteq [n])$ is the canonical basis of $\mathbb{R}^{2^{[n]}}$.

In order to understand which facets intersect, in hope determine the faces of $\mathbb{SC}_n$, the first step is to understand the linear relations between the $\boldsymbol{n}(S,u,v)$. One can show (without too much difficulties) that these linear relations are generated by the *cubal relation*:

$$\boldsymbol{n}(Suvx,u,v) + \boldsymbol{n}(Sux,u,x) = \boldsymbol{n}(Suv,u,v) + \boldsymbol{n}(Suvx,u,x)$$

In particular, using the cubal relations (without using their combinations) is enough to deal with $\mathbb{DC}(\mathsf{Z}_G)$ and $\mathbb{DC}(\mathsf{N}_B)$, the deformation cones of graphical zonotopes and nestohedra. It is probably also enough to deal with deformation cones of quotientopes, but dealing with deformation cones of hypergraphic polytopes require using combinations of cubal relations.

(Semi-)open problem 4.E.5 — [Deformation cones of generalized permutahedra via cubal relations]
Find a way to exploit cubal relations to compute the facets of $\mathbb{DC}(\mathsf{P})$ *for any generalized permutahedron* P.

CHAPTER

5

MONOTONE PATH-, PIVOT-, SWEEP-, SECONDARY- AND FIBER POLYTOPES

Content of the chapter

Section 5.1

Monotone paths, Coherent paths, Monotone path polytope

Definition 5.1.1 — [Linear program]
A linear program is the data of finitely many linear inequalities (the constraints), together with a linear functional to optimize (the objective function). Such a linear program can be represented by a couple $(\mathsf{P}, \boldsymbol{c})$*, where* $\mathsf{P} \subseteq \mathbb{R}^d$ *is the set of solution of the constrains, and the objective function is* $\boldsymbol{x} \mapsto \langle \boldsymbol{x}, \boldsymbol{c}\rangle$ *for* $\boldsymbol{c} \in \mathbb{R}^d$*.*

In general, P *can be an (unbounded) polyhedra or the empty set, but we limit ourself to the study of bounded linear programs, that is to say when* P *is a polytope. We assume this case in all what follows, and let the reader extend the definitions and theorem to unbounded cases.*

Definition 5.1.2 — [Monotone paths]
For a linear program $(\mathsf{P}, \boldsymbol{c})$*, a* $\boldsymbol{c}$*-monotone path is a directed path in the directed graph* $G_{\mathsf{P},\boldsymbol{c}}$ *from one of its sources to one of its sinks. When* $\boldsymbol{c}$ *is clear from the context, we do not precise it.*

Recall that, if $\boldsymbol{c}$ *is generic (i.e. for all edge* $\boldsymbol{uv}$ *of* P*, one has* $\langle \boldsymbol{u}, \boldsymbol{c}\rangle \neq \langle \boldsymbol{v}, \boldsymbol{c}\rangle$*), then the source and sink is unique.*

⋆ **Exercise 5.1.3**
How many monotone paths of length k are there for the linear program $(\Delta_{d-1}, \boldsymbol{c})$ for a generic $\boldsymbol{c}$?

How many monotone paths of length k are there for the linear program $(\square_d, \boldsymbol{c})$ for a generic $\boldsymbol{c} = (1, \ldots, 1)$?

Conjecture 5.1.4 — [Monotone paths on 3-polytopes]
A (simple) 3-polytope with $2n$ *vertices has* $\leq F_{n+2}+1$ *monotone paths, where* F_n *is the* n^{th} *Fibonacci number (*$F_1 = F_2 = 1$*,* $F_{n+2} = F_{n+1} + F_n$*)*[1]*.*

[1] Probably, by the time this course take place, Guyer & I will have solved this conjecture.

⋆ **Exercise 5.1.5**
When was Leonardo Fibonacci born? Around 1170.

We will do a wedge on a polygon: Fix a polygon with $n+1$ vertices in the plane $x_3 = 0$ with an edge in direction $(1, 0, 0)$, make a cylinder over it, in direction $(0, 0, 1)$, and add the inequalities $x_3 \geq 0$ and $x_1 + x_3 \leq 0$. The 3-polytope obtained is a wedge over a $(n+1)$-gon. Show that it has $2n$ vertices.

Show that, for a well chosen starting polygon and by wiggling a bit the coordinates of the vertices of the wedge, one can construct a 3-polytope with $2n$ vertices such that it has $F_{n+2}+1$ monotone paths in a well-chosen direction.

Remark 5.1.6 What follows on monotone path polytope is mostly copy-pasted from one of my papers [Pou24].

Definition 5.1.7 — [Coherent path]
For a linear program $(\mathsf{P}, \boldsymbol{c})$ *and a secondary direction* $\boldsymbol{\omega} \in \mathbb{R}^d$ *(linearly independent to* $\boldsymbol{c}$*), one can consider the polygon* $\mathsf{P}_{\boldsymbol{c},\boldsymbol{\omega}}$ *obtained by projecting* P *onto the plane spanned by* $(\boldsymbol{c}, \boldsymbol{\omega})$*:*

$$\mathsf{P}_{\boldsymbol{c},\boldsymbol{\omega}} := \big\{ \big(\langle \boldsymbol{x}, \boldsymbol{c}\rangle \,,\; \langle \boldsymbol{x}, \boldsymbol{\omega}\rangle\big) \;;\; \boldsymbol{x} \in \mathsf{P} \big\}$$

A proper face (vertex or edge) G *of* $\mathsf{P}_{\boldsymbol{c},\boldsymbol{\omega}}$ *is an upper face if it has an outer normal vector with positive second coordinate*[1]*, equivalently if* $(x_1, x_2) + (0, \varepsilon) \notin \mathsf{P}_{\boldsymbol{c},\boldsymbol{\omega}}$ *for all* $(x_1, x_2) \in \mathsf{G}$*, and* $\varepsilon > 0$*.*

A monotone path $\mathcal{L}$ *is coherent if there exists* $\boldsymbol{\omega} \in \mathbb{R}^d$ *such that* $\mathcal{L}$ *is the family of pre-images by* $\pi_{\boldsymbol{c}}$ *of the lower faces of* $\mathsf{P}_{\boldsymbol{c},\boldsymbol{\omega}}$*. In this case, such an* $\boldsymbol{\omega}$ *is said to capture the coherent path* $\mathcal{L}$*.*

[1] Some definitions in the literature use lower faces, we take upper faces to ease drawings and notations.

Definition 5.1.8 — [Monotone path polytope]
For a linear program $(\mathsf{P}, \boldsymbol{c})$*, consider the projection* $\pi_{\boldsymbol{c}} : \boldsymbol{x} \mapsto \langle \boldsymbol{x}, \boldsymbol{c}\rangle$*. Denoting the image segment* $\mathsf{Q} = \pi_{\boldsymbol{c}}(\mathsf{P}) = \{\langle \boldsymbol{x}, \boldsymbol{c}\rangle \;;\; \boldsymbol{x} \in \mathsf{P}\}$*, The monotone path polytope* $\mathsf{M}_{\boldsymbol{c}}(\mathsf{P})$ *is defined by:*

$$\mathsf{M}_{\boldsymbol{c}}(\mathsf{P}) := \left\{ \int_{\mathsf{Q}} \gamma(x)\mathrm{d}x \;\;;\;\; \gamma \text{ section of } \pi_{\boldsymbol{c}} \right\}$$

Remark 5.1.9 This definition is awful, and everyone should immediately forget about it. We first state a nice theorem explaining why one would want to study the monotone path polytope, and then explain how to construct it.

Theorem 5.1.10 *The vertices of* $\mathsf{M}_{\boldsymbol{c}}(\mathsf{P})$ *are in bijection with coherent monotone paths on* $(\mathsf{P}, \boldsymbol{c})$*.*

The non-coherent monotone paths correspond to (some) interior points of $\mathsf{M}_{\boldsymbol{c}}(\mathsf{P})$*.*

◊ **Construction 5.1.11**
Let's now look at the vertices of the monotone path polytopes $\mathsf{M}_{\boldsymbol{c}}(\mathsf{P})$ and expose this construction more clearly. Figure 5.1 gives an illustration.

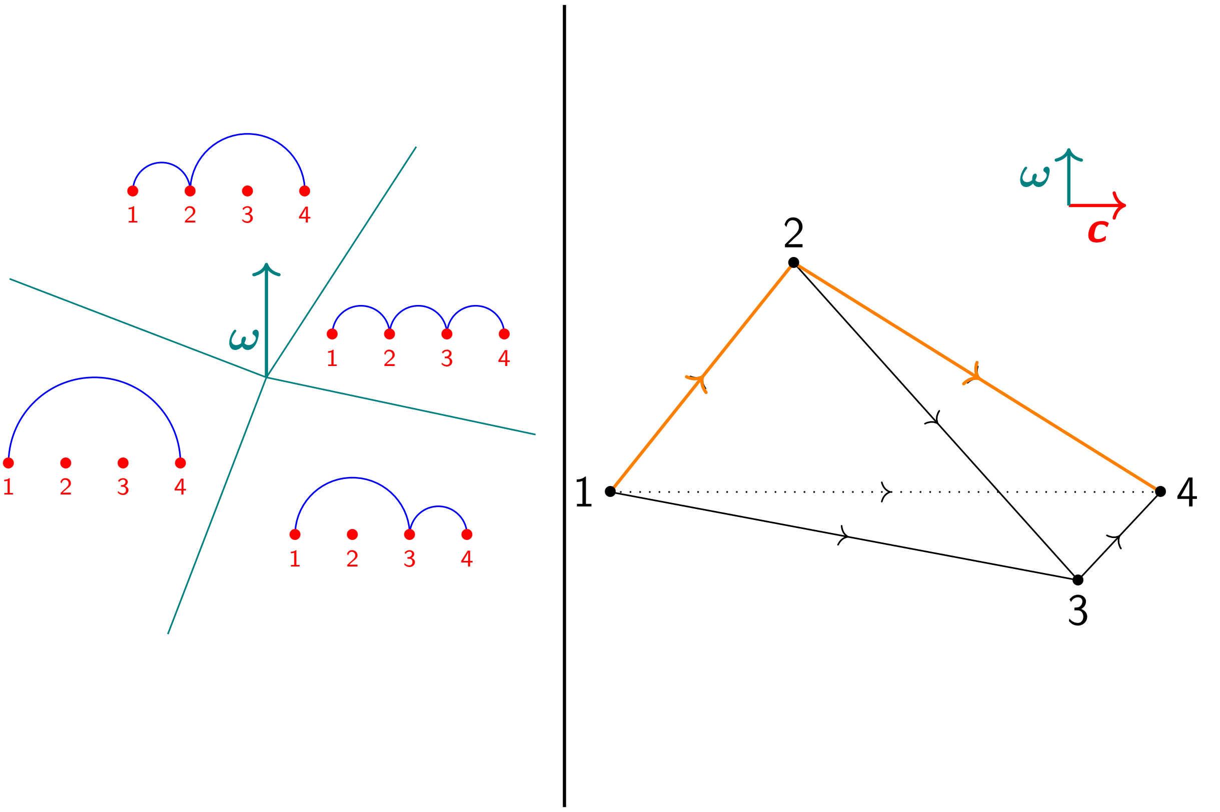

Figure 5.1: Animated construction of the normal fan of the monotone path polytope of the 3-dimensional simplex. For each $\boldsymbol{\omega} \in \mathbb{R}^3$ orthogonal to $\boldsymbol{c}$, we project Δ_3 onto the plane spanned by $(\boldsymbol{c}, \boldsymbol{\omega})$ (Right), and record the corresponding coherent monotone path (Left). (Animated figures obviously do not display on paper, please use a PDF viewer (like Adobe Acrobat Reader), or go on my personal website, or ask by email.)

Fix a polytope $\mathsf{P} \subset \mathbb{R}^d$, and a generic direction $\boldsymbol{c} \in \mathbb{R}^d$.

Consider another direction $\boldsymbol{\omega} \in \mathbb{R}^d$, linearly independent of $\boldsymbol{c}$, and project P onto the plane spanned by $(\boldsymbol{c}, \boldsymbol{\omega})$. In Figure 5.1, we take a tetrahedron and a direction $\boldsymbol{c}$, and then scan through all possible $\boldsymbol{\omega}$ (it is enough to scan only $\boldsymbol{\omega}$ with $\langle \boldsymbol{\omega}, \boldsymbol{c} \rangle = 0$). The tetrahedron is naturally projected onto the plane $(\boldsymbol{c}, \boldsymbol{\omega})$ by looking at the boundary of the drawing (*i.e.* the 3 or 4 outside edges). This polygon has two paths from its minimal (leftmost) vertex to its maximal (rightmost) vertex: a lower one and a upper one. When there exists $\boldsymbol{\omega}$ such that a given monotone path $\mathcal{L}$ is projected onto the upper path (and no 2-face of P projects onto it), then $\mathcal{L}$ is coherent.

The vertices of the monotone path polytope $\mathsf{M}_{\boldsymbol{c}}(\mathsf{P})$ are not in bijection to all the monotone paths on P, but only to the coherent ones. The faces of higher dimension are obtained following the same ideas. In Figure 5.1, we record on the left the (coherent) monotone path obtained for each choice of $\boldsymbol{\omega}$ on the chosen tetrahedron, in this case, all monotone paths are coherent.

We now present four ways to visualize the monotone path polytope: the two firsts focus on its normal fan, while the two lasts allow for an explicit computation of the vertices.

Exploring the space of $\boldsymbol{\omega}$, and projection of the normal fan of P First of all, focus on the space of all $\boldsymbol{\omega}$, and partition it depending on the coherent coherent paths they yield. Precisely, to a coherent path $\mathcal{L}$ we associate $\mathcal{N}(\mathcal{L}) = \{\boldsymbol{\omega} \ ; \ \boldsymbol{\omega} \text{ captures } \mathcal{L}\}$. Then $\mathcal{N}(\mathcal{L})$ is a polyhedral cone by linearity (in $\boldsymbol{\omega}$) of the projection from P onto $\mathsf{P}_{\boldsymbol{c},\boldsymbol{\omega}}$, and the cones $\mathcal{N} = \big(\mathcal{N}(\mathcal{L})\big)_{\mathcal{L}}$ are the maximal cones of a fan. This fan is exactly the normal fan of $\mathsf{M}_{\boldsymbol{c}}(\mathsf{P})$. Hence, one can run through all possible $\boldsymbol{\omega} \in \mathbb{R}^d$, orthogonal to $\boldsymbol{c}$ (as all $\boldsymbol{\omega} + \lambda\boldsymbol{c}$ capture the same coherent path for any $\lambda \in \mathbb{R}$), to draw the normal fan of

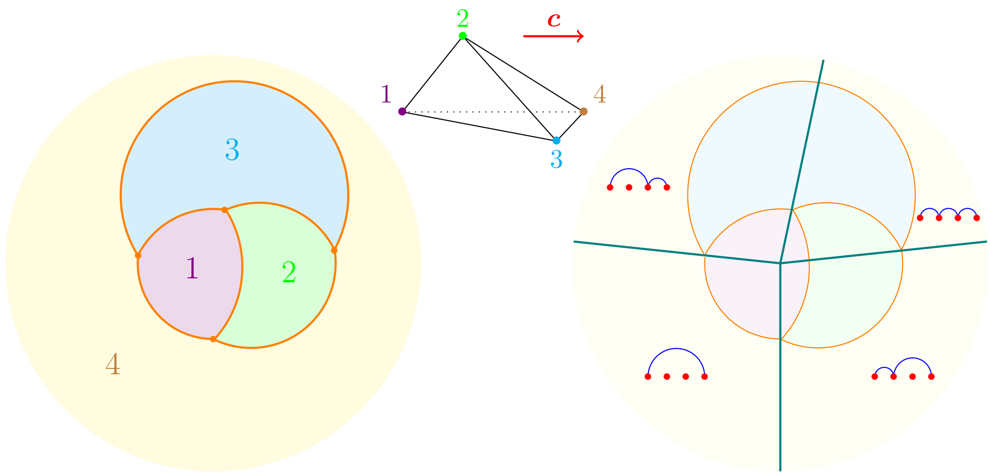

Figure 5.2: (Top) For reference, the tetrahedron $\mathsf{P} = \Delta_3$, and the direction $\boldsymbol{c}$ from Figure 5.1. (Left) The stereographic projection of the normal fan of P, each colored region correspond to (the normal cone of) a vertex of P. (Right) Two rays give rise to the same coherent monotone path if and only they intersect the same colored regions, we draw the resulting fan, labeled accordingly.

$\mathsf{M}_{\boldsymbol{c}}(\mathsf{P})$, see Figure 5.1 for the construction. A given $\boldsymbol{\omega}$ captures the coherent path $(\boldsymbol{v}_0, \dots, \boldsymbol{v}_r)$ such that $\boldsymbol{v}_0 = \boldsymbol{v}_{\min}$, $\boldsymbol{v}_r = \boldsymbol{v}_{\max}$ and $\boldsymbol{v}_i\boldsymbol{v}_{i+1}$ is the edge of P with $\langle \boldsymbol{v}_i, \boldsymbol{c}\rangle < \langle \boldsymbol{v}_{i+1}, \boldsymbol{c}\rangle$ satisfying that $\frac{\langle \boldsymbol{v}_{i+1}-\boldsymbol{v}_i, \boldsymbol{\omega}\rangle}{\langle \boldsymbol{v}_{i+1}-\boldsymbol{v}_i, \boldsymbol{c}\rangle}$ is the unique maximizer of $\frac{\langle \boldsymbol{v}_j-\boldsymbol{v}_i, \boldsymbol{\omega}\rangle}{\langle \boldsymbol{v}_j-\boldsymbol{v}_i, \boldsymbol{c}\rangle}$ for $\boldsymbol{v}_i\boldsymbol{v}_j$ and edge of P with $\langle \boldsymbol{v}_i, \boldsymbol{c}\rangle < \langle \boldsymbol{v}_j, \boldsymbol{c}\rangle$.

This construction shows an important property: for a fixed $\boldsymbol{\omega}$, all $\boldsymbol{\omega} + \lambda\boldsymbol{c}$ for $\lambda \in \mathbb{R}$ capture the same coherent path. Consequently, one can obtain the normal fan of $\mathsf{M}_{\boldsymbol{c}}(\mathsf{P})$ by *projecting* the normal fan of P: to each normal cone $\mathsf{C} \in \mathcal{N}_{\mathsf{P}}$, associate its projection along $\boldsymbol{c}$, namely $\mathsf{C}_\perp := \{\boldsymbol{x} - \frac{\langle \boldsymbol{x}, \boldsymbol{c}\rangle}{\langle \boldsymbol{c}, \boldsymbol{c}\rangle}\boldsymbol{c} ; \boldsymbol{x} \in \mathsf{C}\}$. The common refinement of $\big(\mathsf{C}_\perp ; \mathsf{C} \in \mathcal{N}_{\mathsf{P}}\big)$ is the normal fan[1] of $\mathsf{M}_{\boldsymbol{c}}(\mathsf{P})$.

Line bundle and stereographic projection A second way to visualize the combinatorics of the monotone path polytope $\mathsf{M}_{\boldsymbol{c}}(\mathsf{P})$ is to imagine the lines $\ell_{\boldsymbol{\omega}} := (\boldsymbol{\omega} + \lambda\boldsymbol{c} \ ; \ \lambda \in \mathbb{R})$, and consider the *line bundle* $(\ell_{\boldsymbol{\omega}} \ ; \ \boldsymbol{\omega} \in \mathbb{R}^d)$. Each line $\ell_{\boldsymbol{\omega}}$ intersects the normal fan $\mathcal{N}_{\mathsf{P}}$ of the polytope P, and the cones it intersects describe the coherent path that $\boldsymbol{\omega}$ captures: if $\ell_{\boldsymbol{\omega}}$ intersects only maximal cones and cones of co-dimension 1, then it captures a coherent monotone path (which is the case for almost all $\boldsymbol{\omega}$). Looking at which maximal cones of $\mathcal{N}_{\mathsf{P}}$ are intersected by $\ell_{\boldsymbol{\omega}}$ yields the list of vertices forming the associated coherent monotone path; whereas looking at which co-dimension 1 cones are intersected by $\ell_{\boldsymbol{\omega}}$ yields the list of edges forming the associated coherent monotone path.

To visualize this easily (especially if $\dim \mathsf{P} = 3$), one can use the stereographic projection $\mathrm{st}_{\boldsymbol{c}} : \mathbb{R}^d \to \mathbb{R}^{d-1}$ that maps the apex $\frac{\boldsymbol{c}}{\|\boldsymbol{c}\|}$ to infinity, see Figure 5.2. The normal fan $\mathcal{N}_{\mathsf{P}}$ projects onto a subdivision $\mathrm{st}_{\boldsymbol{c}}(\mathcal{N}_{\mathsf{P}})$ of $\mathbb{R}^{d-1}$ by spherical cap (*i.e.* arcs of circles if $\dim \mathsf{P} = 3$). Besides, the counterpart on the sphere of $\ell_{\boldsymbol{\omega}}$, is the arc $\alpha_{\boldsymbol{\omega}} = \big(\frac{\boldsymbol{\omega}+\lambda\boldsymbol{c}}{\|\boldsymbol{\omega}+\lambda\boldsymbol{c}\|} \ ; \ \lambda \in \mathbb{R}\big)$. This is an arc of a great circle containing the apex $\frac{\boldsymbol{c}}{\|\boldsymbol{c}\|}$ and its antipodal point: thus $\mathrm{st}_{\boldsymbol{c}}(\alpha_{\boldsymbol{\omega}})$ is ray (from $\boldsymbol{0} \in \mathbb{R}^{d-1}$ to infinity). The cells of $\mathrm{st}_{\boldsymbol{c}}(\mathcal{N}_{\mathsf{P}})$ that the ray $\mathrm{st}_{\boldsymbol{c}}(\alpha_{\boldsymbol{\omega}})$ intersects are the cones of $\mathcal{N}_{\mathsf{P}}$ that $\ell_{\boldsymbol{\omega}}$ intersects, and hence describe the coherent path that $\boldsymbol{\omega}$ captures: by looking how all rays of $\mathbb{R}^{d-1}$ intersect $\mathrm{st}_{\boldsymbol{c}}(\mathcal{N}_{\mathsf{P}})$, we get a drawing of the normal fan of $\mathsf{M}_{\boldsymbol{c}}(\mathsf{P})$ in $\mathbb{R}^{d-1}$, see Figure 5.2 (Right).

This point of view can come in handy when one wants to vary $\boldsymbol{c}$. Indeed, varying $\boldsymbol{c}$ amounts to varying the apex of the stereographic projection, *i.e.* to "roll" the projection $\mathrm{st}_{\boldsymbol{c}}(\mathcal{N}_{\mathsf{P}})$ inside $\mathbb{R}^{d-1}$. This "rolling" is hard to describe, but at least, the rays we want to intersect it with remain fixed.

Convex hull of (explicit) points A third way to construct the monotone path polytope $\mathsf{M}_{\boldsymbol{c}}(\mathsf{P})$ is to use the following formula from [BS92, Theorem 5.3]. Let $V(\mathsf{P}) = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ with $\langle \boldsymbol{v}_i, \boldsymbol{c}\rangle \leq \langle \boldsymbol{v}_j, \boldsymbol{c}\rangle$ for $i \leq j$. For a monotone path $\mathcal{L} = (\boldsymbol{v}_{i_1}, \dots, \boldsymbol{v}_{i_r})$ on P (with $i_1 = 1$ and $i_r = n$), denote

$$\psi(\mathcal{L}) = \sum_{j=1}^{r} \frac{\langle \boldsymbol{v}_{i_j} - \boldsymbol{v}_{i_{j-1}}, \boldsymbol{c}\rangle}{2\,\langle \boldsymbol{v}_n - \boldsymbol{v}_1, \boldsymbol{c}\rangle}\,(\boldsymbol{v}_{i_j} + \boldsymbol{v}_{i_{j-1}})$$

Then $\mathsf{M}_{\boldsymbol{c}}(\mathsf{P}) = \mathrm{conv}\big(\psi(\mathcal{L}) \ ; \ \text{for } \mathcal{L} \text{ monotone path on } \mathsf{P}\big)$. A point $\psi(\mathcal{L})$ is a vertex of $\mathsf{M}_{\boldsymbol{c}}(\mathsf{P})$ if and only if $\mathcal{L}$ is a coherent monotone path.

For the case of the simplex, all monotone paths are coherent, so a figure would not be very enlightening. We picture a better example in Figure 5.3: (Left) is drawn a 3-dimensional polytope, (Right) its monotone path polytope, obtained via the above formula. The two red crosses correspond to non-coherent monotone paths $\mathcal{L}$, for which the point $\psi(\mathcal{L})$ lie inside $\mathsf{M}_{\boldsymbol{c}}(\mathsf{P})$.

[1] This construction embeds the fan $\mathcal{N}_{\mathsf{M}_{\boldsymbol{c}}(\mathsf{P})}$ directly into the hyperplane $\boldsymbol{c}^\perp$, instead of embedding it in $\mathbb{R}^{\dim(\mathsf{P})}$.

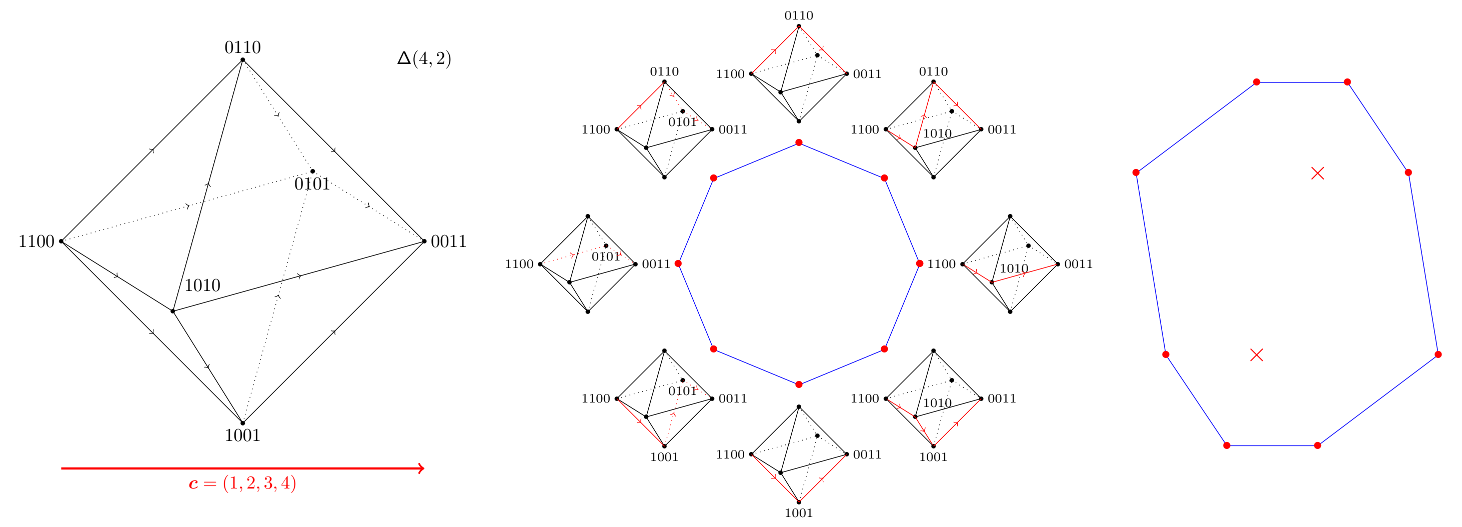

Figure 5.3: (Left) The $(4,2)$-hypersimplex lives in the hyperplane $\{\boldsymbol{x} \,;\, \sum_{i=1}^{4} x_i = 2\}$ inside $\mathbb{R}^4$. (Middle) Its monotone path polytope is an octagon, each vertex of which is labeled by the corresponding (coherent) monotone path, drawn on $\Delta(4,2)$. (Right) Actually, this monotone path is not a *regular* octagon, but the octagon depicted here, the two crosses correspond to the two monotone paths on $\Delta(4,2)$ which are not coherent.

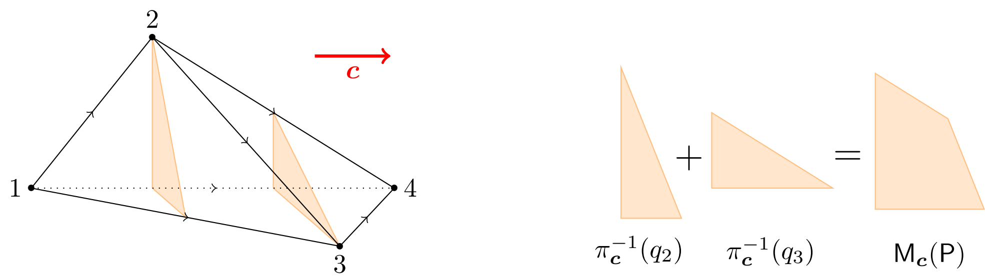

Figure 5.4: The construction of $\mathsf{M}_{\boldsymbol{c}}(\mathsf{P})$ as a sum of sections for the tetrahedron $\mathsf{P} = \Delta_3$. Each section is orthogonal to $\boldsymbol{c}$ and contains a vertex (except for $\boldsymbol{v}_{\min}$ and $\boldsymbol{v}_{\max}$).

Minkowski sum of sections A forth way to visualize monotone path polytopes is to use [BS92, Theorem 1.5] which provides a re-writing of the integral as a finite Minkowski sum. This sum is constructed as follows. We begin by sorting the vertices of P according to their scalar product against $\boldsymbol{c}$: $V(\mathsf{P}) = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ with $\langle \boldsymbol{v}_i, \boldsymbol{c}\rangle \leq \langle \boldsymbol{v}_{i+1}, \boldsymbol{c}\rangle$. The segment $\mathsf{Q} = \pi_{\boldsymbol{c}}(\mathsf{P})$ is cut out by the projection into sub-segments $\mathsf{C}_i := [q_i, q_{i+1}]$ with $q_i = \langle \boldsymbol{v}_i, \boldsymbol{c}\rangle$, and the barycenter (*i.e.* middle) of C_i is trivially $b_i = \frac{q_i + q_{i+1}}{2}$. The monotone path polytope $\mathsf{M}_{\boldsymbol{c}}(\mathsf{P})$ is normally equivalent to the Minkowski sum of sections $\sum_{i=1}^{n} \pi_{\boldsymbol{c}}^{-1}(b_i)$.

Though exact, this construction is a bit unhandy. Yet, as we will prove in **??**, one can forget about centers, as $\mathsf{M}_{\boldsymbol{c}}(\mathsf{P})$ is normally equivalent to $\sum_{i=2}^{n-1} \pi_{\boldsymbol{c}}^{-1}(q_i)$. This gives beautiful pictures, see Figure 5.4 for the case of the tetrahedron.

Note that, between the figures, a slight change of perspective happened. The fans constructed in Figure 5.1 (Left) and Figure 5.2 (Right) are the same, and are the normal fan of the $\mathsf{M}_{\boldsymbol{c}}(\mathsf{P})$ appearing in Figure 5.4 (Right), even thought a right angle *seems* to appear on the latter but not on the firsts.

Theorem 5.1.12 *For a linear program* $(\mathsf{P}, \boldsymbol{c})$ *with* $\boldsymbol{c}$ *generic, let* $V(\mathsf{P}) = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$, *and* $q_i = \langle \boldsymbol{v}_i, \boldsymbol{c}\rangle$ *with* $q_1 \leq \dots \leq q_n$. *The monotone path polytope* $\mathsf{M}_{\boldsymbol{c}}(\mathsf{P})$ *is normally equivalent to the Minkowski sum of sections* $\sum_{i=2}^{n-1} \{\boldsymbol{x} \in \mathsf{P} \,;\, \langle \boldsymbol{x}, \boldsymbol{c}\rangle = q_i\}$.

⋆ **Exercise 5.1.13**

Let $\mathsf{P}_i = \{\boldsymbol{x} \in \mathsf{P} \,;\, \langle \boldsymbol{x}, \boldsymbol{c}\rangle = q_i\}$, and $\mathsf{P}_{i,i+1} = \{\boldsymbol{x} \in \mathsf{P} \,;\, q_i \leq \langle \boldsymbol{x}, \boldsymbol{c}\rangle \leq q_{i+1}\}$. Show $\mathsf{P}_{i,i+1}$ is the Cayley polytope $\mathsf{Cay}(\mathsf{P}_i, \mathsf{P}_{i+1})$.

Deduce that the section $\{\boldsymbol{x} \in \mathsf{P} \,;\, \langle \boldsymbol{x}, \boldsymbol{c}\rangle = \frac{q_i + q_{i+1}}{2}\}$ can be written as a Minkowski sum of P_i and P_{i+1}.

Conclude that $\mathsf{M}_{\boldsymbol{c}}(\mathsf{P}) = \sum_i \mathsf{P}_i$, and explain why we can remove P_1 and P_n from this sum.

⋆ **Exercise 5.1.14 — [Billera–Strumfels, 1992]**

Prove that the monotone path polytope of a simplex (for a generic direction) is a cube, and hence that all the monotone paths on a simplex are coherent. Prove that the monotone path polytope of a cube for $\boldsymbol{c} = (1, \dots, 1)$ is a permutahedron, and hence that all the monotone paths on a cube are coherent.

(Semi-)open problem 5.1.15 — [Monotone path polytope of the permutahedron]
Compute (the number of vertices of) the monotone path polytope of the permutahedron for $\boldsymbol{c} = (1, 2, \dots, n)$.

Section 5.2

Triangulations, Secondary polytopes

Definition 5.2.1 — [Simplicial complex (abstract and not)]
A simplicial complex $K = (\mathsf{S}_1, \dots, \mathsf{S}_k)$ is a collection of simplices in $\mathbb{R}^d$ such that:

- *If $\mathsf{S} \in K$, then all the faces of S are in K.*
- *If $\mathsf{S}, \mathsf{S}' \in K$, then $\mathsf{S} \cap \mathsf{S}'$ is a face of both S and S'.*

An abstract simplicial complex $\mathcal{K}$ is a collection of subsets of $[n]$ such that if $X \in \mathcal{K}$ and $Y \subseteq X$, then $Y \in \mathcal{K}$. Some authors may consider $\varnothing \notin \mathcal{K}$.

A simplicial complex K realizes an abstract simplicial complex $\mathcal{K}$ if $\mathcal{K} = (X \ ; \ \mathrm{conv}\{\boldsymbol{v}_i \ ; \ i \in X\} \in K)$ where $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ are the vertices of K (i.e. the vertices of its simplices).

⋆ **Exercise 5.2.2**
Show that every abstract simplicial complex $\mathcal{K}$ on n vertices can be realized. Use the faces of the standard simplex of dimension n.

Definition 5.2.3 — [Dimension, Pure, Carrier, Vertex set of (abstract) simplicial complexes]
The dimension of a simplex X in an (abstract) simplicial complex $\mathcal{K}$ is $\dim X = |X| + 1$.

A (abstract) simplicial complex is pure if all its inclusion-maximal simplices have the same dimension.

The vertex set of a simplicial complex is the union of the sets of vertices of the simplices it contains.

The carrier (or underlying space) of a simplicial complex is the union of its simplices.

Remark 5.2.4 Simplicial complexes, abstract or not, are fundamental objects in many fields of mathematics, from algebraic topology to pure combinatorics, as well as computational geometry. We introduce it here so that the reader starts to get at ease with the notion, but we will not get into the details.

Definition 5.2.5 — [Point configuration]
A point configuration is a finite collection of points $\boldsymbol{A} = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_r)$, where $\boldsymbol{a}_i \in \mathbb{R}^d$.

Sometimes, the points could be allowed with repetitions (and $\boldsymbol{A}$ could be a multi-set), but not in this course.

Definition 5.2.6 — [Triangulation]
A triangulation of a point configuration $\boldsymbol{A}$ is a simplicial complex T whose carrier is $\mathrm{conv}\,\boldsymbol{A}$, and whose vertex set is included in $\boldsymbol{A}$. Abstractly, a triangulation should be thought as the abstract simplicial complex: $\mathcal{T} := (X \ ; \ \mathrm{conv}\{\boldsymbol{a}_i \ ; \ i \in X\} \in T)$, such that for every extremal vertex $\boldsymbol{a}_i \in \boldsymbol{A}$, then i is a vertex of $\mathcal{T}$.

A triangulation T is full if its vertex set is equal to $\boldsymbol{A}$. Abstractly, a triangulation $\mathcal{T}$ is full if for point $\boldsymbol{a}_i \in \boldsymbol{A}$, then i is a vertex of $\mathcal{T}$.

The word "triangulation of a polytope $\mathsf{P} \subset \mathbb{R}^d$" can either refer to a (full) triangulation of its vertices $V(\mathsf{P})$, or to a full triangulation of all its lattice points $\mathsf{P} \cap \mathbb{Z}^d$.

⋆ **Exercise 5.2.7**
There are 3 possible points configurations of 4 points: for each of them, compute all the triangulations. Which ones are full?

⋆ **Exercise 5.2.8**
How many triangulations of a n-gon are there? How many are full?

Definition 5.2.9 — [Unimodular simplices, Unimodular triangulation]
A simplex $\mathsf{S} = \mathrm{conv}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_{d+1})$ is unimodular if it is a lattice simplex (i.e. all its vertices have integer coordinates) and its volume is $\frac{1}{d!}$, that is to say if: $\det\left(\begin{pmatrix}1\\ \boldsymbol{v}_1\end{pmatrix}, \dots, \begin{pmatrix}1\\ \boldsymbol{v}_{d+1}\end{pmatrix}\right) = 1$.

A triangulation is unimodular if all its simplices are.

⋆ **Exercise 5.2.10**
Show that $\frac{1}{d!}$ is the minimal volume of a simplex of dimension d with integer coordinates.

⋆ **Exercise 5.2.11**
Show that if P has a unimodular triangulation $\mathcal{T}$ (using either its vertices or its lattice points), then its $\mathrm{vol}\,\mathsf{P} = |\mathcal{T}|$, where vol is the lattice volume (i.e. $\mathrm{vol}\,\Delta_{d-1} = 1$).

⋆ **Exercise 5.2.12 — [Reeve simplex]**

Consider Reeve's simplex: $\mathsf{Reeve}_q = \mathrm{conv}\left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}q\\1\\1\end{pmatrix}\right)$ for an integer $q > 0$.

Compute the volume of Reeve_q, and show it does not have a unimodular triangulation using its lattice points, for $q > 1$.

Theorem 5.2.13 — [Knudsen, 1977, Haase–Paffenholz–Piechnik–Santos, 2014]
If P *is a lattice* d*-polytopes (i.e. all its vertices have integer coordinates), then there exists an integer* $k \geq 0$ *such that* $k\mathsf{P}$ *has a unimodular triangulation. Moreover,* $k \leq (d+1)!^{(\operatorname{vol}\mathsf{P})!(d+1)^{(d+1)^2 \operatorname{vol}\mathsf{P}}}$*, where* vol *is the lattice volume (i.e.* $\operatorname{vol}\Delta_{d-1} = 1$*).*

Remark 5.2.14 See the proof in [HPPS14].

Remark 5.2.15 Unimodular triangulations are of prime importance for computing volumes, but also for the study of Ehrhart theory, and has deep link with algebraic combinatorics. We mention it here for the sake of completeness and to help the reader understand the other courses she of he might attend, but we focus on regular triangulations.

Definition 5.2.16 — [Lower faces, Lower facets]
For a polytope $\mathsf{P} \subset \mathbb{R}^d$*, a face* F *is a lower face if there exists* $\boldsymbol{c}$ *satisfying* $\mathsf{F} = \mathsf{P}^{\boldsymbol{c}}$ *and* $c_d < 0$*.*
Especially, F *is a lower facet is the last coordinate of its (outer) normal is strictly negative.*

Definition 5.2.17 — [Regular triangulation]
A triangulation $\mathcal{T}$ *of a point configuration* $\boldsymbol{A}$ *is regular if there exists a height function given by a vector* $\boldsymbol{\omega} \in \mathbb{R}^{\boldsymbol{A}}$*, such that the lower faces of* $\mathsf{P}^{\boldsymbol{A},\boldsymbol{\omega}} := \operatorname{conv}\left(\begin{pmatrix}\boldsymbol{a}_i \\ \omega_{\boldsymbol{a}_i}\end{pmatrix} \in \mathbb{R}^{d+1} \;;\; \boldsymbol{a}_i \in \boldsymbol{A}\right)$ *are* $\operatorname{conv}\left(\begin{pmatrix}\boldsymbol{a}_i \\ \omega_{\boldsymbol{a}_i}\end{pmatrix} \in \mathbb{R}^{d+1} \;;\; i \in X\right)$ *for* $X \in \mathcal{T}$*.*
We say that $\boldsymbol{\omega}$ *captures the triangulation* $\mathcal{T}$*.*

⋆ **Exercise 5.2.18**
Prove that any point configuration has (at least) one triangulation, moreover a regular one.

⋆ **Exercise 5.2.19**
Show that the two triangulations of a quadrangle are regular.

⋆ **Exercise 5.2.20 — [A non-regular triangulation]**
Suppose two adjacent vertices of a quadrangle are at height 0, write the inequality in the height of the two other vertices which governs which triangulation is captured by this height function.

In the plane $\mathbb{R}^2$, consider a triangle $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ with barycenter $\mathbf{0}$, and its scaled version $\boldsymbol{a}_4 = \lambda\boldsymbol{a}_1$, $\boldsymbol{a}_5 = \lambda\boldsymbol{a}_2$, $\boldsymbol{a}_6 = \lambda\boldsymbol{a}_3$ with $\lambda > 1$. Show that the triangulation $\mathcal{T} = (123, 124, 235, 136, 146, 245, 356)$ is not regular.

⋆ **Exercise 5.2.21 — [Order triangulation of the cube]**
For a permutation $\sigma \in S_n$, the *k-th prefix* is the set $\sigma_{\leq k} = \{\sigma(1), \dots, \sigma(k)\}$.

The *order simplex* Δ_σ associated to a permutation σ is the simplex $\operatorname{conv}(\boldsymbol{e}_{\sigma_{\leq k}} \;;\; 0 \leq k \leq n)$ where $\boldsymbol{e}_X = \sum_{i \in X} \boldsymbol{e}_i$. The *order triangulation* (also *Kuhn's triangulation*, also *staircase triangulation*) is the triangulation of the cube $\square_n$ whose maximal simplices are the order simplices $(\Delta_\sigma \;;\; \sigma \in S_n)$.

Show that the order triangulation is a triangulation of $\square_n$, which is unimodular (hint: $\operatorname{vol}\square_n = n!$), and regular (hint: take $\omega_X = -|X|^2$). Show that the simplices in the order triangulation come in two classes of rotation-equivalent simplices. Show that these classes are equivalent by central symmetry.

The *dual graph* $G_{\mathcal{T}}$ of a triangulation $\mathcal{T}$ is the graph whose node set is $\mathcal{T}$ and where two simplices $X, X' \in \mathcal{T}$ are linked by an arc if $|X \cap X'| = |X| - 1$.

Show that the order triangulation is induced by cutting the cube by the braid arrangement. Deduce that the dual graph of this triangulation is the graph of the permutahedron.

⋆ **Exercise 5.2.22 — [Haiman triangulation of the cube, 1991]**
First, find a triangulation of the 3-cube into 5 tetrahedra.

Now, consider the product of simplices $\Delta_{d-1} \times \Delta_{n-1} \subset \mathbb{R}^{d+n}$. The vertices of $\Delta_{d-1} \times \Delta_{n-1}$ can be associated to the arcs of the complete bipartite graph $K_{d,n}$ on $[d] \sqcup [n]$. A simplex S in $\Delta_{d-1} \times \Delta_{n-1}$ is associated to a subgraph of $K_{d,n}$.

Show that such a simplex S is full dimensional if and only if the associated sub-graph is a tree.

Deduce that all full-dimensional simplices inside $\Delta_{d-1} \times \Delta_{n-1}$ are unimodular.

How many simplices are there in a triangulation of $\Delta_{d-1} \times \Delta_{n-1}$?

If a d-polytope P is triangulated into r simplices, and a n-polytope Q into s simplices, construct a triangulation of $\mathsf{P} \times \mathsf{Q}$ into $rs\binom{n+d}{d}$ simplices.

Deduce that if $\square_n$ can be triangulated into t_n simplices, then $\square_{kn}$ can be triangulated into $\left(\frac{t_n}{n!}\right)^k (kn)!$.

Conclude that, asymptotically, $\square_n$ can be triangulated using only $\rho^n\, n!$ simplices, with $\rho = \left(\frac{5}{6}\right)^{\frac{1}{3}} \simeq 0.941$.

Show that one can require this triangulation to be regular.

Remark 5.2.23 Using "small" triangulations of $\square_8$ obtained by Sallee (1984), one can improve to $\rho = \left(\frac{13\,248}{40\,320}\right)^{\frac{1}{8}} \simeq 0.870$.

(Semi-)open problem 5.2.24 — [Small (regular) triangulations of the cube]
Find a (regular) triangulation of the cube $\square_n$ *using the minimum number of simplices, especially for* $n \geq 8$*. Try to get closer to the asymptotic lower bound:* $O(c^n\sqrt{n!})$ *for some constant number* c*.*

Definition 5.2.25 — [Secondary fan]
The secondary fan of a point configuration $\boldsymbol{A}$ is the fan in $\mathbb{R}^{\boldsymbol{A}}$ where $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ belong to the same cone if they capture the same (regular) triangulation.

For a (regular) triangulation, we denote $\mathcal{N}_{sec}(\mathcal{T}) = \{\boldsymbol{\omega} \in \mathbb{R}^{\boldsymbol{A}} \; ; \; \boldsymbol{\omega}$ captures $\mathcal{T}$ on $\boldsymbol{A}\}$. By convention $\mathcal{N}_{sec}(\mathcal{T}) = \varnothing$ for non-regular triangulations of $\boldsymbol{A}$. Hence $\mathcal{N}_{sec} = \big(\mathcal{N}_{sec}(\mathcal{T}) \; ; \; \mathcal{T}$ triangulation of $\boldsymbol{A}\big)$.

⋆ Exercise 5.2.26
Show that the secondary fan is a complete pointed fan. It is essential?

Definition 5.2.27 — [Gel'fand–Kapranov–Zelevinsky vector]
The Gel'fand–Kapranov–Zelevinsky vector $\boldsymbol{\Psi}(\mathcal{T}) \in \mathbb{R}^{\boldsymbol{A}}$ of a triangulation $\mathcal{T}$ of a point configuration $\boldsymbol{A}$ is given by its coordinate on the point $\boldsymbol{a}_i \in \boldsymbol{A}$:

$$\boldsymbol{\Psi}(\mathcal{T})_{\boldsymbol{a}_i} = \sum_{i \in X \in \mathcal{T}} \mathrm{vol}\big(\mathrm{conv}(\boldsymbol{a}_j \; ; \; j \in X)\big)$$

⋆ Exercise 5.2.28
To see if you are at ease with the notations, show that: $\boldsymbol{\Psi}(\mathcal{T}) = \sum_{\mathsf{S} \in \mathcal{T}} (\mathrm{vol}\,\mathsf{S})\, \boldsymbol{e}_{V(\mathsf{S})}$.

Definition 5.2.29 — [Secondary polytope]
The secondary polytope $\Sigma(\boldsymbol{A})$ of a point configuration $\boldsymbol{A}$ is the convex hull of the Gel'fand–Kapranov–Zelevinsky vectors of its triangulations: $\Sigma(\boldsymbol{A}) = \mathrm{conv}(\boldsymbol{\Psi}(\mathcal{T}) \; ; \; \mathcal{T}$ triangulation of $\boldsymbol{A})$.

Theorem 5.2.30 — [Gel'fand–Kapranov–Zelevinsky, 1991]
The normal fan of the secondary polytope $\Sigma(\boldsymbol{A})$ is the secondary fan.

Especially, the vertices of the secondary polytope correspond to regular triangulations, while the Gel'fand–Kapranov–Zelevinsky vector of non-regular triangulations lies in the interior of the secondary polytope $\Sigma(\boldsymbol{A})$.

Moreover, $\mathcal{N}_{\Sigma(\boldsymbol{A})}(\boldsymbol{\Psi}(\mathcal{T})) = \mathcal{N}_{sec}(\mathcal{T})$.

⋆ Exercise 5.2.31
Prove this theorem.

(Semi-)open problem 5.2.32 — [Secondary polytope of the cube]
Compute the secondary polytope of the cube $\square_n$ for $n \geq 4$.

Note that, $\Sigma(\square_3)$ is fully known: it has dimension 5 and 74 vertices. However, $\Sigma(\square_4)$ has dimension 12 and 87 959 448 vertices, but is not fully known. For $n \geq 5$, the number of vertices is not known, and $\dim \Sigma(\square_n) = 2^n - n$.

(Semi-)open problem 5.2.33 — [Triangulations of random polytopes]
Study triangulations and secondary polytope of random polytopes.

For instance, construct the convex hull of n points taken at random on a 4-sphere, and look a the number of **regular** *triangulations* vs *the number of triangulations. Does this ratio goes to 0 when $n \to +\infty$?*

⋆ Exercise 5.2.34 — [The secondart polytope of polygon is an associahedron]
Consider the cyclic n-gon: $\mathsf{Cyc}(2,n) = \mathrm{conv}\big((i,i^2) \; ; \; 1 \leq i \leq n\big)$ (it works for any n-gon, but it is easier with the cyclic one).

Fix a height function $\boldsymbol{\omega} \in R^n$ with associated triangulation $\mathcal{T}$. Let $ijk\ell$ be a quandrangle of $\mathcal{T}$, *i.e.* the union of two adjacent triangles. Show that the triangulation of the quadrangle $ijk\ell$ depends on the sign of: $\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ i & j & k & \ell \\ i^2 & j^2 & k^2 & \ell^2 \\ \omega_i & \omega_j & \omega_k & \omega_\ell \end{pmatrix}$.

For $\omega_x = \sum_{q=0}^{n} \alpha_q x^q$ for some $\alpha_q \in \mathbb{R}$, show that this determinant is a polynomial of degree n in the variables i, j, k and ℓ.
Using Lagrange interpolation, deduce that, by choosing the α_q wisely, one can prescribe the sign of these determinants.
Deduce that all triangulations of the cyclic n-gon are regular.

Deduce that the graph of the secondary polytope $\Sigma(\mathsf{Cyc}(2,n))$ is the one of an associahedron, and thus that $\Sigma(\mathsf{Cyc}(2,n))$ is an associahedron.

Denoting $\boldsymbol{v}_i = (i, i^2)$, show that $\mathrm{vol}(\boldsymbol{v}_i\boldsymbol{v}_j\boldsymbol{v}_k) = \frac{1}{2}(j-i)(k-i)(k-j)$.
Deduce that the vertex of the secondary polytope $\Sigma(\mathsf{Cyc}(2,n))$ associated to the triangulation $\mathcal{T}$ is:

$$\boldsymbol{\Psi}(\mathcal{T}) = \frac{1}{2} \sum_{ijk \in \mathcal{T}} (j-i)(k-i)(k-j)\,(\boldsymbol{e}_i + \boldsymbol{e}_j + \boldsymbol{e}_k)$$

Deduce that $\Sigma(\mathsf{Cyc}(2,n))$ is not normally equivalent to Loday's associahedron.

Remark 5.2.35 Triangulations is a vaaaaaaast subject, and there are are many subjects that we could have talked about. A non-exhaustive list a key words that the interested reader might look for is: Delaunay triangulations, simplices with integer decomposition property, Ehrhart theory, triangulations of order polytopes, hyperplan induced triangulation, flag triangulation, Cayley trick, Gröbner bases, ...

We recommend the excellent book of [LRS10], adorned by 550 figures!

Section 5.3

Fiber polytopes

The following is mainly an extract of my thesis, re-adapted for the purpose of this course.

We give a very brief introduction to fiber polytopes, secondary polytopes and π-coherent subdivisions arising from a polytope projection $\pi : \mathsf{P} \to \mathsf{Q}$. For an instructive and illustrated presentation of the subject, we advise the reader to look at [Zie95, Chapter 9], a more in depth explanation can be found in [ALRS00, Section 2] and [LRS10, Chapter 9.1], and the original articles [BS92] (for fiber polytopes) and [GKZ90, GKZ91] (for secondary polytopes) give the details of the proofs.

Definition 5.3.1 — [Polytope projection]
A polytope projection is a couple (P, π) where $\mathsf{P} \subset \mathbb{R}^d$ is a polytope and $\pi : \mathbb{R}^d \to \mathbb{R}^{d'}$ is a projection. When dimensions are obvious or irrelevant, we usually denote such a projection by $\pi : \mathsf{P} \to \mathsf{Q}$ assuming that $\mathsf{Q} := \pi(\mathsf{P})$.

Definition 5.3.2 — [Polyhedral complex, Polyhedral subdivision]
A polyhedral complex $\mathcal{C}$ is a collection of polytopes such that if $\mathsf{P} \in \mathcal{C}$, then all the faces of P are in $\mathcal{C}$, and if $\mathsf{P}, \mathsf{Q} \in \mathcal{C}$, then the intersection $\mathsf{P} \cap \mathsf{Q}$ is a face of both P and Q.

A subdivision of a polytope Q is a polyhedral complex $\mathcal{C}$ such that $\bigcup_{\mathsf{P}\in\mathcal{C}} \mathsf{P} = \mathsf{Q}$.

Definition 5.3.3 — [Induced subdivision, Baues poset]
For a polytope projection $\pi : \mathsf{P} \to \mathsf{Q}$, a π-induced subdivision of Q is a subdivision $\pi(\mathcal{F})$ of Q where:

- *$\pi(\mathcal{F}) = \{\pi(F) \; ; \; F \in \mathcal{F}\}$ for $\mathcal{F}$ a family of faces of P.*
- *for $F, F' \in \mathcal{F}$, if $\pi(F) \subseteq \pi(F')$, then $F = F' \cap \pi^{-1}(\pi(F))$.*

The set of π-induced subdivisions is ordered by refinement, forming the Baues poset: $\pi(\mathcal{F}_1) \preccurlyeq \pi(\mathcal{F}_2)$ when every polytope of $\pi(\mathcal{F}_2)$ is a union of polytopes of $\pi(\mathcal{F}_1)$. More conveniently, as $\mathcal{F}$ can be recovered from the knowledge of $\pi(\mathcal{F})$ (see [Zie95, Chapter 9]), one has that $\pi(\mathcal{F}_1) \preccurlyeq \pi(\mathcal{F}_2)$ if and only if $\bigcup_{F\in\mathcal{F}_1} F \subseteq \bigcup_{F\in\mathcal{F}_2} F$.

By convention, the empty family will be considered a π-induced subdivision. It is the minimal element of the Baues poset. Note that even if they are called *subdivisions*, the π-induced subdivisions are better thought of not as subdivisions of Q, but as polyhedral complexes that live in P (and whose projection by π is a subdivision of Q). Among π-induced subdivisions, some appear as special (regular) subdivisions, we follow here the reformulation of [Zie95].

Definition 5.3.4 — [Coherent subdivision]
Let $\pi : \mathsf{P} \to \mathsf{Q}$ be a polytope projection with $\dim \mathsf{P} = d$ and $\dim \mathsf{Q} = d'$. For $\boldsymbol{\omega} \in \mathbb{R}^d$, define $\pi^{\boldsymbol{\omega}} : \mathbb{R}^d \to \mathbb{R}^{d'+1}$ by

$$\pi^{\boldsymbol{\omega}}(\boldsymbol{x}) = \begin{pmatrix} \pi(\boldsymbol{x}) \\ \langle \boldsymbol{\omega}, \boldsymbol{x} \rangle \end{pmatrix}$$

The family of lower faces of $\pi^{\boldsymbol{\omega}}(\mathsf{P})$ projects down to Q by forgetting the last coordinate, giving rise to a π-induced subdivision of Q. The π-induced subdivisions of this form are called π-coherent subdivisions, and form a sub-poset of the Baues poset: the lattice of π-coherent subdivisions.

We say that $\boldsymbol{\omega}$ captures the subdivision.

Note that when $\boldsymbol{\omega}$ is generic with respect to P, then the associated π-coherent subdivision is a finest π-coherent subdivision in the sense that it covers the empty subdivision in the Baues poset.

The fiber polytope has several (equivalent) definitions. Here, even though the formal definition is given here, we will not use the realization of the fiber polytope, but only focus on the characterization of its face lattice given in the following Theorem 5.3.6.

Definition 5.3.5 — [Fiber polytope, integral definition]
For a polytope projection $\pi : \mathsf{P} \to \mathsf{Q}$, a section of P is a continuous map $\gamma : \mathsf{Q} \to \mathsf{P}$ satisfying $\pi \circ \gamma = id_{\mathsf{Q}}$. The fiber polytope $\Sigma_\pi(\mathsf{P}, \mathsf{Q})$ for the projection $\pi : \mathsf{P} \to \mathsf{Q}$ is defined by:

$$\Sigma_\pi(\mathsf{P}, \mathsf{Q}) = \left\{ \frac{1}{vol(\mathsf{Q})} \int_{\mathsf{Q}} \gamma(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \; ; \; \gamma \text{ section of } \mathsf{P} \right\}$$

Theorem 5.3.6 — [[BS92, Corollary 1.4]]
For a polytope projection $\pi : \mathsf{P} \to \mathsf{Q}$, the fiber polytope $\Sigma_\pi(\mathsf{P}, \mathsf{Q})$ is a polytope and its face lattice is (isomorphic to) the lattice of π-coherent subdivisions of Q.

⋆ Exercise 5.3.7
Show that $\Sigma_\pi(\mathsf{P}, \mathsf{Q})$ is of dimension $\dim(\mathsf{P}) - \dim(\mathsf{Q})$, though embedded in $\mathbb{R}^{\dim(\mathsf{P})}$.

The construction of fiber polytopes through Definition 5.3.5 is cumbersome for numerical computations and drawings. Fortunately, the following theorem provides a description of fiber polytopes as a **finite** Minkowski sum.

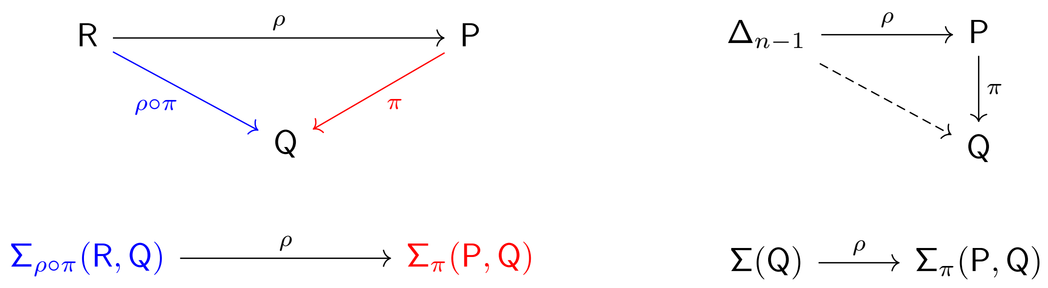

Figure 5.5: (Left) A projection $\rho : \mathsf{R} \to \mathsf{P}$ induces a projection between the fiber polytopes of R and P for their projections onto Q. Note that as ρ and π are projection $|V(\mathsf{R})| \geq |V(\mathsf{P})| \geq |V(\mathsf{Q})|$. (Right) If $n = |V(\mathsf{Q})|$, then $\Sigma_\pi(\mathsf{P}, \mathsf{Q})$ is a projection of $\Sigma(\mathsf{Q})$ when $|V(\mathsf{P})| = |V(\mathsf{Q})| = n$.

Theorem 5.3.8 — [Fiber polytope, finite Minkowski sum definition - [BS92, Theorem 1.5]]
For the polytope projection $\pi : \mathsf{P} \to \mathsf{Q}$, consider the subdivision of Q defined as the common refinement of all $\pi(\mathsf{F})$ for F a face of P. For each maximal cell C of this subdivision, we denote $\boldsymbol{b}_\mathsf{C}$ the barycenter (or centroid) of C. Then:

$$\Sigma_\pi(\mathsf{P}, \mathsf{Q}) = \frac{1}{\mathrm{vol}(\mathsf{Q})} \sum_{\mathsf{C} \text{ maximal cells}} \mathrm{vol}(\mathsf{C})\ \pi^{-1}(\boldsymbol{b}_\mathsf{C})$$

Even though an adequate construction of a category of polytopes is still lacking, fiber polytopes have a categorical flavor. Indeed, if one would construct a category **Pol** in which objects are polytopes, and morphisms are (surjective) projections between polytopes, then the map $(\pi : \mathsf{P} \to \mathsf{Q}) \mapsto \Sigma_\pi(\mathsf{P}, \mathsf{Q})$ would resemble a functor from the category of morphisms of **Pol** to **Pol** itself. The commutative diagram of Figure 5.5(Left) indicates how the (categorical) cone over Q would be sent to **Pol** by this functor. Notably, the following theorem guaranties fiber polytopes are well-behaved with respect to projections:

Theorem 5.3.9 — [[BS92, Lemma 2.3]]
For two polytopes projections $\rho : \mathsf{R} \to \mathsf{P}$ and $\pi : \mathsf{P} \to \mathsf{Q}$, one has: $\Sigma_\pi(\mathsf{P}, \mathsf{Q}) = \rho\big(\Sigma_{\pi\circ\rho}(\mathsf{P}, \mathsf{R})\big)$.

⋆ **Exercise 5.3.10**
Prove this theorem.

⋆ **Exercise 5.3.11 — [Trivial fiber polytopes]**
Show that $\Sigma_\pi(\mathsf{P}, \mathbf{0}) = \mathsf{P}$. Show that $\Sigma_\pi(\mathsf{P}, \mathsf{P})$ is a point.

⋆ **Exercise 5.3.12 — [Monotone path polytopes are fiber polytopes]**
For $(\mathsf{P}, \boldsymbol{c})$, show that the monotone path polytope $\mathsf{M}_{\boldsymbol{c}}(\mathsf{P})$ is normally equivalent to $\Sigma_\pi(\mathsf{P}, \mathsf{Q})$ where $\pi : \begin{array}{ccc} \mathbb{R}^d & \to & \mathbb{R} \\ \boldsymbol{x} & \mapsto & \langle \boldsymbol{x}, \boldsymbol{c} \rangle \end{array}$.

⋆ **Exercise 5.3.13 — [Secondary polytopes are fiber polytopes]**
Show that $\Sigma(\mathsf{P})$ and $\Sigma_\pi(\Delta_n, \mathsf{P})$ are normally equivalent. One actually even has: $\Sigma(\mathsf{P}) = (d+1)\mathrm{vol}(\mathsf{P})\,\Sigma_\pi(\Delta_n, \mathsf{P})$.

Corollary 5.3.14 — [Fiber polytopes are projections of secondary polytopes]
For $\pi : \mathsf{P} \to \mathsf{Q}$, let $\boldsymbol{A} = \pi(V(\mathsf{P}))$ be the point configuration obtained by projecting the vertices of P. Then $\Sigma_\pi(\mathsf{P}, \mathsf{Q})$ arises as a projection of the secondary polytope of $\boldsymbol{A}$, i.e. there exists a projection ρ such that: $\Sigma_\pi(\mathsf{P}, \mathsf{Q}) = \rho\big(\Sigma(\boldsymbol{A})\big)$.
Especially, if $|V(\mathsf{P})| = |V(\mathsf{Q})|$, then: $\Sigma_\pi(\mathsf{P}, \mathsf{Q}) = \rho\big(\Sigma(\mathsf{Q})\big)$.

⋆ **Exercise 5.3.15**
Prove this corollary by taking ρ to be the usual projection from the standard simplex $\Delta_{|V(\mathsf{P})|}$ to the P.

⋆ **Exercise 5.3.16 — [Monotone path polytopes are not zonotopes]**
Prove that monotone path polytopes are projections of cubes. Yet, explain why monotone path polytopes **are not** zonotopes!

⋆ **Exercise 5.3.17**
Define of *(coherent) cellular strings*, the counterpart of coherent monotone paths for the faces of monotone path polytopes.

⋆ **Exercise 5.3.18**
Define of *(regular) coarsest subdivisions*, the counterpart of regular triangulations for the facets of secondary polytopes.

(Semi-)open problem 5.3.19 — [Fiber polytopes]
Compute the fiber polytope of almost anything (except between cyclic polytopes), especially for projections onto 2- or 3-dimensional polytopes, or for polytopes with a strong combinatorial flavor.

APPENDICE 5.D

Monotone arborescences, Coherent arborescences, Pivot polytopes

Definition 5.D.1 — [Monotone arborescence]
For a linear program $(\mathsf{P}, \boldsymbol{c})$, a $\boldsymbol{c}$-monotone arborescence is a map $A : V(\mathsf{P}) \to V(\mathsf{P})$ such that $\boldsymbol{v}A(\boldsymbol{v})$ is an edge of P with $\langle \boldsymbol{v}, \boldsymbol{c} \rangle < \langle A(\boldsymbol{v}), \boldsymbol{c} \rangle$.
By convention, $A(\boldsymbol{v}_{\max}) = \boldsymbol{v}_{\max}$.

⋆ **Exercise 5.D.2 — [Athanasiadis, 2021]**
Show that a linear program $(\mathsf{P}, \boldsymbol{c})$ has $\prod_{\boldsymbol{v} \neq \boldsymbol{v}_{\max}} d_{\text{out}}(\boldsymbol{v})$ monotone arborescences, where the out-degree is taken in the directed graph $G_{\mathsf{P},\boldsymbol{c}}$.

Definition 5.D.3 — [Coherent arborescence]
For a linear program $(\mathsf{P}, \boldsymbol{c})$, a $\boldsymbol{c}$-monotone arborescence is coherent, if there exists $\boldsymbol{\omega} \in \mathbb{R}^d$ such that $A(\boldsymbol{v})$ is the unique maximizer of $\frac{\langle \boldsymbol{u}-\boldsymbol{v}, \boldsymbol{\omega} \rangle}{\langle \boldsymbol{u}-\boldsymbol{v}, \boldsymbol{c} \rangle}$ for $\boldsymbol{v}\boldsymbol{u}$ an edge of P satisfying $\langle \boldsymbol{v}, \boldsymbol{c} \rangle < \langle \boldsymbol{u}, \boldsymbol{c} \rangle$.
We say that $\boldsymbol{\omega}$ captures this coherent arborescence.

⋆ **Exercise 5.D.4**
Give a geometric interpretation to the coherence of a monotone arborescence (make a drawing in the plane, as for coherent monotone paths).

Definition 5.D.5 — [Pivot fan]
The (max-slope) pivot fan is the fan in which $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ belong to the same maximal cone if they capture the same arborescence.

Theorem 5.D.6 — [Black–De Loera–Lütjeharms–Sanyal, 2022]
There exists a polytope, called the (max-slope) pivot polytope whose normal fan is the (max-slope) pivot fan. It is the convex of $\sum_{\boldsymbol{v} \neq \boldsymbol{v}_{\max}} \frac{1}{\langle A(\boldsymbol{v}) - \boldsymbol{v}, \boldsymbol{c} \rangle} (A(\boldsymbol{v}) - \boldsymbol{v})$, for A a $\boldsymbol{c}$-monotone arborescence of $(\mathsf{P}, \boldsymbol{c})$.

⋆ **Exercise 5.D.7**
Prove that the (max-slope) pivot polytope is normally equivalent to the Minkowski sum of the following sections: for each vertex $\boldsymbol{v}$ of P, take $\mathsf{P}_{\boldsymbol{v}}$ the convex hull of the edges $\boldsymbol{v}\boldsymbol{u}$ satisfying $\langle \boldsymbol{v}, \boldsymbol{c} \rangle < \langle \boldsymbol{u}, \boldsymbol{c} \rangle$, intersect $\mathsf{P}_{\boldsymbol{v}}$ with an hyperplane $H_{\boldsymbol{c}, \langle \boldsymbol{v}, \boldsymbol{c} \rangle + \varepsilon}$ for a small $\varepsilon > 0$.

⋆ **Exercise 5.D.8**
Show that the monotone path polytope of $(\mathsf{P}, \boldsymbol{c})$ is a deformation of the (max-slope) pivot polytope of $(\mathsf{P}, \boldsymbol{c})$.

Theorem 5.D.9 — [Black–Lütjeharms–Sanyal, 2024]
The (max-slope) pivot polytope of a simplex (for a generic direction) is an associahedron.

Theorem 5.D.10 — [Pilaud–P., 2024]
The (max-slope) pivot polytope of products of simplices (for a generic direction) is are shuffles of associahedra.
We do not define shuffles here: it is an operation on generalized permutahedra which yields a generalized permutahedron.

(Semi-)open problem 5.D.11 — [Pivot polytopes which are generalized permutahedra]
Are the only (max-slope) pivot polytopes which are generalized permutahedra, the ones of product of simplices?
Are (max-slope) pivot polytopes projections of generalized permutahedra?

Theorem 5.D.12 *Let $\pi : \mathsf{P} \to \mathsf{Q}$ be a projection between polytopes. If $G_{\mathsf{P}} = G_{\mathsf{Q}}$, then the (max-slope) pivot polytope of Q is the projection by π of the (max-slope) pivot polytope of P.*

Remark 5.D.13 This is far weaker than the case of fiber polytopes! Indeed, recall that projections do, usually, delete vertices, so the graph is not very rarely retained.

⋆ **Exercise 5.D.14**
Show that the pivot polytope of a cyclic polytope $\mathsf{Cyc}(d, n)$ with $d \geq 4$ is a projection of an associahedron.

APPENDICE 5.E

Sweeps, Sweep polytope

Definition 5.E.1 — [Sweep]
For a point configuration $\boldsymbol{A} = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_n) \subset \mathbb{R}^d$, *and* $\boldsymbol{\omega} \in \mathbb{R}^d$, *the* $\boldsymbol{\omega}$*-sweep is the ordered partition* $(B_1, \dots, B_r)$ *of* $[n]$ *given by the ordering* $i \leq j$ *if and only if* $\langle \boldsymbol{a}_i, \boldsymbol{\omega} \rangle \leq \langle \boldsymbol{a}_j, \boldsymbol{\omega} \rangle$, *i.e.* i *and* j *are in the same* B_k *if* $\langle \boldsymbol{a}_i, \boldsymbol{\omega} \rangle = \langle \boldsymbol{a}_j, \boldsymbol{\omega} \rangle$, *and* $i \in B_k$, $j \in B_p$ *with* $k < p$ *if* $\langle \boldsymbol{a}_i, \boldsymbol{\omega} \rangle < \langle \boldsymbol{a}_j, \boldsymbol{\omega} \rangle$.

We say that $\boldsymbol{\omega}$ *captures this sweep.*

These ordered partition are ordered by refinement to create the lattice of sweeps.

Definition 5.E.2 — [Sweep fan]
The sweep fan is the fan in which $\boldsymbol{\omega}$ *and* $\boldsymbol{\omega}'$ *belong to the same maximal cone if they capture the same sweep.*

Theorem 5.E.3 — [Padrol–Philippe, 2023]
There exists a polytope, called the sweep polytope, whose normal fan is the sweep fan. It is obtained as the zonotope $\sum_{1 \leq i < j \leq n} \left[-\frac{1}{2}(\boldsymbol{a}_i - \boldsymbol{a}_j),\ \frac{1}{2}(\boldsymbol{a}_i - \boldsymbol{a}_j) \right]$.

⋆ Exercise 5.E.4
Show that the sweep polytope of a simplex is the permutahedron.

⋆ Exercise 5.E.5
Show that the sweep polytope is a projection of the permutahedron Π_n.

Show that the sweep fan is induced by an hyperplane arrangement.

⋆ Exercise 5.E.6
Show that the sweep polytope is the monotone path polytope of the zonotope $\frac{n}{2} \sum_{i \in [n]} \left[-\begin{pmatrix} \boldsymbol{a}_i \\ 1 \end{pmatrix},\ \begin{pmatrix} \boldsymbol{a}_i \\ 1 \end{pmatrix} \right]$, for $\boldsymbol{c} = \boldsymbol{e}_{n+1}$.

Remark 5.E.7 Conversely, monotone path polytopes of zonotopes for generic $\boldsymbol{c}$ are sweep polytopes, up to normal equivalence.

(Semi-)open problem 5.E.8 — [Sweep polytopes of cyclic polytopes, Castillo–Labbé, 2024]
Describe the sweep polytope of (the vertices of) cyclic polytopes. Especially, what is the number $N(d, n)$ *of sweeps of the* d*-dimensional cyclic polytope with* n*-vertices? Is there a closed formula when* $n \ll d$*?*

CHAPTER

6

EXAM

Content of the chapter

Section 6.1

Questionnaire (multiple choice)

⋆ **Exercise 6.1.1**
What is a convex polytope?
1) The convex hull of a collection of points.
2) The feasible domain of a linear program.
3) The intersection of finitely many closed half-spaces.
4) A bounded convex set whose boundary has curvature 0 almost everywhere.

⋆ **Exercise 6.1.2**
Which of the following implies that the polytopes P and Q have the same face lattice?
1) The graph of P and the graph of Q are isomorphic.
2) There is a bijection between the facets of P and the facets of Q that sends a facet to one with the same face lattice.
3) For all k, the number of faces of dimension k of P is the same as the one of Q.
4) There exists a affine isomorphism (i.e. affine bijective transformation) L with L(P) = Q.

⋆ **Exercise 6.1.3**
Sort these polytopes according to their number of vertices (for d large).
d-cube, d-simplex, d-permutahedron, d-associahedron

⋆ **Exercise 6.1.4**
Which one is true in dimension 3 but false in dimension 4?
1) If two polytopes have the same graph, they have the same face lattice.
2) For every polytope P, there exists Q with the same face lattice and all the vertices of Q have integer coordinates.
3) In every facets F of a polytope, the edge vectors sum to zero (i.e. there exists a way to orient the edges $\boldsymbol{u} \to \boldsymbol{v}$ such that $\sum_{\boldsymbol{u}\to\boldsymbol{v}\in E(\mathsf{F})} \boldsymbol{v} - \boldsymbol{u} = \boldsymbol{0}$).
4) Every lattice polytope (i.e. vertices with integer coordinates) has a dilate that can be triangulated by lattice simplices of minimal volume (i.e. volume $\frac{1}{d!}$).

⋆ **Exercise 6.1.5**
A Minkowski sum of 2 co-planar triangles can have: 3, 4, 5 or 6 edges?

Section 6.2

Graph Associahedra

⋆ Exercise 6.2.1

Recall: for a set X, we denote $\boldsymbol{e}_X := \sum_{i\in X} \boldsymbol{e}_i$, and $\Delta_X = \mathrm{conv}(\boldsymbol{e}_i \; ; \; i \in X)$.

In a connected graph $G = (V, E)$, a *tube* is a connected induced sub-graph (considered as a sub-set of the vertices of G), except that V itself is not considered a tube. The *graph associahedron* of G is defined as $\mathsf{Asso}_G := \sum_{t \text{ tube of } G} \Delta_t$.

1) Show that Asso_{P_n} is the usual associahedron (of Loday), where P_n is the path on n vertices.

2) Show that Asso_{K_n} is (normally equivalent to) the permutahedron, where K_n is the complete graph on n vertices.

3) Show that for any graph G, then Asso_G is a generalized permutahedron (*a.k.a.* a deformed permutahedron).

4) Prove that $\dim \mathsf{Asso}_G = |V| - 1$. (Hint1: what's $\sum_i x_i$? / Hint2: edges are tubes)

5) Fix a tube t. What is the nomal fan of Δ_t, in particular its rays?

6) Deduce the rays of the normal fan of Asso_G (carefull:low dimension stuffs may behave weirdly): show that Asso_G has one facet per tube of G.

7) Give a (irredundant) inequality description of Asso_G.

8) A *tubing* is a collection of tubes $\mathcal{T} = (t_1, \dots, t_r)$ such that for all $i \neq j$, then either $t_i \subsetneq t_j$, or $t_j \subseteq t_i$, or there is no edge between t_i and t_j (*i.e.* $t_i \cup t_j$ is **not** a tube). Show that the vertices of Asso_G are in bijection with maximal tubings (and its faces with tubings). [Hardest question]

9) Show that a maximal tubing has $n - 1$ tubes. Deduce that Asso_G is a simple polytope.

10) Deduce the dimension of the deformation cone of Asso_G.

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