Mathematics

Oriented matroids

Short introduction

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Contents

CHAPTER

1

BASIC DEFINITIONS

Content of the chapter

Attention 1.0.1 *Recall that:*

$\mathbb{R}^d$ is a vector space of finite dimension d. Usually, the vectors are denoted in bold $\boldsymbol{v} \in \mathbb{R}^d$. The canonical basis of $\mathbb{R}^d$ is denoted $\boldsymbol{e}_1, \dots, \boldsymbol{e}_d$, and for $X \subseteq [n]$ we write $\boldsymbol{e}_X := \sum_{i\in X} \boldsymbol{e}_i$. The space $\mathbb{R}^d$ is endowed with a scalar product denoted $\langle \boldsymbol{u}, \boldsymbol{v}\rangle := \sum_{i=1}^d u_i v_i \in \mathbb{R}$ for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d$. In reality, almost all we do (especially the computations) will be done in $\mathbb{Q}^d$ or even $\mathbb{Z}^d$, but we still use $\mathbb{R}^d$ in definitions and theorems.

Notation 1.0.2 *We use the word "vector" when we think about a linear problem, "point" when we think about affine geometry, "direction" when we think about a vector in the dual.*

We denote $[n] = \{1, 2, \dots, n\}$.

Attention 1.0.3 *The aim of the exercise is to understand the notions at stake. None of them will be graded (except if we need to for administrative reasons). If you manage to do an exercise without making a nice drawing, then you should re-do it! More generally, this course contains very few drawings, the aim being that you (i.e. the reader, the learner) make your own drawings.*

Most of the proofs of the theorem claimed will be done in exercises.

World	Tools	Notion
topology	homology & triangulation	simplicial complexes
alignements	point & lines	matroids
orientations	chirotope	oriented matroids
(geo)metry	scalar product	Euclidian spaces

Section 1.1

Basic notions of matroids

⋆ Exercise 1.1.1
How many "ways" are there to put 4 points in the plane? To put 5 points?

Definition 1.1.2
A vector configuration $\boldsymbol{V} = (\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$ is a collection of vectors in $\mathbb{R}^d$.
A point configuration $\boldsymbol{A} = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)$ is a collection of points in $\mathbb{R}^d$.

Definition 1.1.3 — [Vector configuration, point configuration, independent vectors/points]
For a vector configuration $\boldsymbol{V}$, a subset $S \subseteq [n]$ is independent if $\dim \mathrm{Span}(\boldsymbol{v}_i \ ; \ i \in S) = |S|+1$ (i.e. the vectors $(\boldsymbol{v}_i \ ; \ i \in S)$ are linearly independent). Otherwise, S is dependent.
For a point configuration $\boldsymbol{A}$, the same definition hold with the affine dimension (instead of linear).

⋆ Exercise 1.1.4
Draw a regular octahedron, and consider the point configuration $\boldsymbol{A}$ formed by its vertices.
How many points are there? What is the f-vector of the simplicial complex supported on this point configuration?
How many independent sets are there?
If S is independent, and $S' \subseteq S$, is S' independent?
What is the possibile size of a inclusion-wise maximal independent set?
How many maximal independent sets are there?
If S, S' are independent, with $|S| < |S'|$, show that there exists $x \in S' \smallsetminus S$ such that $S \cup \{x\}$ is independent.

Definition 1.1.5 — [Exchange axiom, Independent system]
A collection of subsets $\mathcal{I} \in 2^{[n]}$ satisfies the exchange axiom if:

$$\text{for all } S, S' \in \mathcal{I}, \text{ if } |S| < |S'|, \text{ then there exists } x \in S' \smallsetminus S \text{ such that } S \cup \{x\} \in \mathcal{I}$$

A non-empty simplicial complex satisfying the exchange axiom is called an independence system.

⋆ Exercise 1.1.6
Write clearly the 3 axioms of independent systems.

⋆ Exercise 1.1.7
Show that the r-skeleton of an independence system is an independence system, for any $r \geq 0$.

⋆ Exercise 1.1.8
Show that $\mathcal{I}$ is an independence system if and only if it is a non-empty simplicial complex such that for every $X \subseteq [n]$, the restriction $\mathcal{I}|_X = \{S \in \mathcal{I} \ ; \ S \subseteq X\}$ is a pure simplicial complex.

Show that $\mathcal{I}$ is an independence system if and only if it is a non-empty pure simplicial complex such that for every permutation σ of $[n]$, the lexicographic order (on facets) is a shelling of $\sigma(\mathcal{I}) = \{\sigma(S) \ ; \ S \in \mathcal{I}\}$.

Use one of these characterisation to prove that the simplicial complex on $[4]$ with facets $\{124, 235, 245\}$ is not an independence system.

⋆ Exercise 1.1.9 — [Independence of vectors/points]
Show that the collection $\mathcal{I}$ of independent subsets of a point or vector configuration is an independence system.

⋆ Exercise 1.1.10 — [Independence of sub-forests]
Let G be a graph. Show that the collection $\mathcal{I}$ of all its sub-forests (here "sub" means subset of edges of G) forms an independence system.

⋆ Exercise 1.1.11 — [Transversal]
Let G be a bipartite graph on $[n] \sqcup F$. Let $\mathcal{I} = ([n] \cap M \ ; \ M \text{ matching in } G)$. Show that $\mathcal{I}$ is an independence system.

⋆ Exercise 1.1.12 — [Algebraic independence]
Let $\mathcal{V} = \{f_1, \dots, f_n\} \in \mathbb{F}(X_1, \dots, X_n)$ be a collection of rational functions in n variables. Let $\mathcal{I}$ be the collection of subsets S for which $(f_i \ ; \ i \in S)$ are algebraically independent over $\mathbb{F}$. Show that $\mathcal{I}$ is a independence system.
Example: $\mathcal{V} = (x, xy, y, y^2 - 1, \frac{x}{y}, z^2 + 2) \subseteq \mathbb{F}(x, y, z)$

Definition 1.1.13 — [Linear, affine, graphical, tranversal and algebraic independence systems]
Let $\mathcal{I} \in 2^{[n]}$ be an independence system.
If there exists a vector configuration $\boldsymbol{V} = (\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$ such that $S \in \mathcal{I}$ if and only if $(\boldsymbol{v}_i \ ; \ i \in S)$ is linearly independent, then $\mathcal{I}$ is called a linear independence system.
If there exists a point configuration $\boldsymbol{A} = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)$ such that $S \in \mathcal{I}$ if and only if $(\boldsymbol{a}_i \ ; \ i \in S)$ is affinely independent, then $\mathcal{I}$ is called a affine independence system.

If there exists a graph G on n edges such that $S \in \mathcal{I}$ if and only if the sub-graph $G[S]$ is a forest, then $\mathcal{I}$ is called a graphical independence system.
If there exists a bipartite graph G on $[n] \sqcup F$ such that $S \in \mathcal{I}$ if and only if there exists a matching in G whose endpoints in $[n]$ is S, then $\mathcal{I}$ is called a transversal independence system.
If there exists a collection of elements $\mathcal{V} = (f_1, \dots, f_n)$ in a extension fiels of $\mathbb{F}$ such that $S \in \mathcal{I}$ if and only if $(f_i \; ; \; i \in S)$ is algebraically independent, then $\mathcal{I}$ is called a algebraic independence system.

⋆ Exercise 1.1.14 — [Homogenization]
The *homogenization* of a point configuration $\boldsymbol{A}$ is the vector configuration $\left(\begin{pmatrix}\boldsymbol{a}\\1\end{pmatrix} \; ; \; \boldsymbol{a} \in \boldsymbol{A}\right)$. The *de-homogenization* is defined accordingly.

Show that every linear independence system is also an affine independence system, and reciprocally.

Definition 1.1.15 — [Basis]
A basis of a independence system $\mathcal{I}$ is an inclusion-wise maximal subset $B \in \mathcal{I}$.

Theorem 1.1.16 *All bases of a independence system $\mathcal{I}$ have the same cardinality.*

⋆ Exercise 1.1.17
Prove this theorem (using the exchance axiom).

⋆ Exercise 1.1.18
Prove that $\mathcal{B}$ is the collection of bases of an independence system if and only if the following two axioms hold:

B1. $\mathcal{B} \neq \varnothing$

B2. For all $B, B' \in \mathcal{B}$ and $x \in B \smallsetminus B'$, there exists $y \in B' \smallsetminus B$ such that $(B \smallsetminus \{x\}) \cup \{y\} \in \mathcal{B}$.

Definition 1.1.19 — [System of bases]
A collection of subsets $\mathcal{B} \in 2^{[n]}$ satisfying the above (B1) and (B2) is called a system of bases.

⋆ Exercise 1.1.20
Show that the bases of a independence system are its facets.

Crypto-morphism 1.1.21 *There is a natural bijection between independence systems and systems of bases given by the map $\mathcal{I} \mapsto \{B \in \mathcal{I} \; ; \; |B| = \max_{S \in \mathcal{I}} |S|\}$.*

⋆ Exercise 1.1.22
Write the reciprocal bijection.

⋆ Exercise 1.1.23
What are the bases of a linear/affine independence system?
What are the bases of a graphical independence system?
What are the bases of a transversal independence system?
What are the bases of a algebraic independence system (called *transcendance bases*)?

Definition 1.1.24 — [Dependence axiom, Vectorial system]
A collection of subsets $\mathcal{D} \in 2^{[n]}$ satifies the dependence axiom if

For all $D \neq D' \in \mathcal{D}$, either $D \cap D' \in \mathcal{D}$ or for all $e \in D \cap D'$, $(D \cup D') \smallsetminus \{e\} \in \mathcal{D}$.

A (possibly empty) set system which is closed by taking upper-sets and satisfies the dependence axiom is called a vectorial system, and its elements are called vectors (sorry for the notation, it will be confusing with "vectors" of a vector configuration...). The inclusion-wise minimal elements of a dependence system are called its circuits.

⋆ Exercise 1.1.25
Show that $\mathcal{C}$ is the collection of circuits of a dependence system if and only if it satisfies:

C1. $\varnothing \notin \mathcal{C}$

C2. $C, C' \in \mathcal{C}$ and $C \subseteq C'$ implies $C = C'$

C3. (Circuit elimination) for all $C, C' \in \mathcal{C}$ with $C \neq C'$, and all $e \in C \cap C'$, there exists $\overline{C} \in \mathcal{C}$ satisfying $\overline{C} \subseteq (C \cup C') \smallsetminus \{e\}$

Definition 1.1.26 — [Circuits]
A collection of subsets $\mathcal{C} \in 2^{[n]}$ satisfying the axioms (C1), (C2) and (C3) is called a system of circuits.

⋆ Exercise 1.1.27
Is every circuit in a system of circuits of the same size? (Hint: Take a graphical example.)

Crypto-morphism 1.1.28 *There is a natural bijection between independence systems and dependence systems, given by the map* $\mathcal{I} \mapsto \{[n] \setminus S \; ; \; S \in \mathcal{I}\}$.

⋆ Exercise 1.1.29
Prove this bijection.

Definition 1.1.30 — [Matroid]
The previous bijection implies that there exists "natural" bijections between independence systems, dependence systems, system of bases, and system of circuits. These bijections form the crypto-morphism of matroids.

The data of any of these set systems is called a "matroid". To be precise, a matroid is given by a couple $(E, \mathcal{X})$ *where* E *is the ground set (usually* $E = [n]$*), and* $\mathcal{X}$ *is a set system from which one can (uniquely) recover an independence system (e.g.* $\mathcal{X}$ *can be an independence system, a dependence system, a system of bases, a system of circuits, a rank function, a lattice of flats,...).*

Accordingly, for a matroid $\mathcal{M}$*, we will speak about its independent sets* $\mathcal{I}_\mathcal{M}$*, its bases* $\mathcal{B}_\mathcal{M}$*, its dependent sets* $\mathcal{D}_\mathcal{M}$*, its circuits* $\mathcal{C}_\mathcal{M}$*, its rank function* $\mathrm{rk}_\mathcal{M}$*, its lattice of flats* $\mathcal{F}_\mathcal{M}$*,...*

A matroid is a linear/affine/graphical/transversal/algebraic matroid if its independence system is.

⋆ Exercise 1.1.31
What are the dependent sets and the circuits of a linear/affine/graphical/transversal/algebraic matroid?

⋆ Exercise 1.1.32 — [Several octahedra]
If I say "Let $\mathcal{M}$ be **the** affine matroid obtained from the vertices of an octahedron", is $\mathcal{M}$ well-defined ?

If I say "let Δ be **the** simplicial complex which is the boundary of an octahedron", is Δ well-defined?

⋆ Exercise 1.1.33 — [Connected matroids]
A matroid is *connected* if every two elements lie in a common circuit of $\mathcal{M}$.

How many matroids are there on [3], on [4], on [5]? How many are connected?

Show that the graphical matroid of G is connected if and only if G is 2-connected.

Is the graphical matroid of a path a connected matroid?

Definition 1.1.34 — [Direct sum]
The direct sum of two matroids $\mathcal{M}_1$ *(on ground set* E*) and* $\mathcal{M}_2$ *(on ground set* F*) is the matroid* $\mathcal{M}_1 \oplus \mathcal{M}_2$ *on the ground set* $E \sqcup F$ *whose independence system is* $\mathcal{I}_{\mathcal{M}_1} \times \mathcal{I}_{\mathcal{M}_2}$.

⋆ Exercise 1.1.35
Show that the direct sum of two matroids is a matroid.

Show that the direct sum of linear/graphical matroids is linear/graphical (exhibit an vector configuration/a graph).

Show that a direct sum of two (non-trivial) matroid is not connected. Acutally, the converse hold: connected matroids are the one that cannot be splitted into a (non-trivial) direct sum.

Definition 1.1.36 — [Rank function, closure operator, lattice of flats]
For a matroid $\mathcal{M}$ *on the ground set* E*, its rank function* $\mathrm{rk}_\mathcal{M} : 2^E \to \mathbb{N}$ *is defined* ***on any subset*** $X \in 2^E$ *by:*

$$\mathrm{rk}_\mathcal{M}(X) = \max(|S| \; ; \; S \in \mathcal{I}_\mathcal{M}, \, S \subseteq X)$$

The rank of a matroid if $\mathrm{rk}_\mathcal{M}(E)$.
The closure operator $\mathrm{cl}_\mathcal{M} : 2^E \to 2^E$ *is defined* ***on any subset*** $X \in 2^E$ *by:*

$$\mathrm{cl}_\mathcal{M}(X) = \big\{e \in E \; ; \; \mathrm{rk}_\mathcal{M}(X \cup \{e\}) = \mathrm{rk}_\mathcal{M}(X)\big\}$$

For a matroid $\mathcal{M}$ *on a ground set* E*, a subset* $F \subseteq E$ *is a flat if it is closed, i.e. if* $\mathrm{cl}_\mathcal{M}(F) = F$*. The flats are ordered by inclusion to form the lattice of flats.*

⋆ Exercise 1.1.37
Prove that $F \subseteq E$ is a flat if and only if for all $e \notin F$, we have $\mathrm{rk}_\mathcal{M}(F \cup \{e\}) > \mathrm{rk}_\mathcal{M}(F)$.

⋆ Exercise 1.1.38
What is the lattice of flats of a regular octahedron?
What is the lattice of flats of a generic octahedron (*i.e.* a octahedron with as many different flats as possible)?

♢ Construction 1.1.39
Just for your knowledge, one can define a matroid via its rank function, its closure operator, or its lattice of flats, that is to say that there are a list of axioms for rank functions, a list for (matroidal) closure operator, and a list for lattice of flats:

- Axioms for rank function:

  R1. $0 \leq \mathrm{rk}(X) \leq |X|$

R2. If $X \subseteq Y$, then $\mathrm{rk}(X) \le \mathrm{rk}(Y)$

R3. (Semi-modularity) $\mathrm{rk}(X \cup Y) + \mathrm{rk}(X \cap Y) \le \mathrm{rk}(X) + \mathrm{rk}(Y)$

Comment: Bijection towards independence system: $\mathrm{rk} \mapsto \{S \subseteq E \ ; \ \mathrm{rk}(S) = |S|\}$

- Matroidal closure operator:

CL1. $X \subseteq \mathrm{cl}(X)$

CL2. $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$

CL3. If $X \subseteq Y$, then $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$

CL4. (Exchange) For all $x, y \in E$ and $X \subseteq E$, if $y \in \mathrm{cl}(X \cup \{x\}) \setminus \mathrm{cl}(X)$, then $x \in \mathrm{cl}(A \cup \{y\})$.

Comment: The three first axioms define a closure operator in general (you can check that for $X \subseteq \mathbb{R}^d$, the map $X \mapsto \mathrm{conv}(X)$ is a closure operator, for instance).

- Theorem [Birkhoff]: A partially ordered set $\mathscr{L}$ is the attice of flats of a matroid without loop nor parallel elements if and only if it is an atomic (upper)-semimodular lattice (*a.k.a.* a geometric lattice).

Definition 1.1.40 — [Dual matroid]
For a matroid $\mathcal{M}$, its dual matroid $\mathcal{M}^*$ is the matroid on the same ground set as $\mathcal{M}$, and whose bases are the complements of the bases of $\mathcal{M}$.

⋆ **Exercise 1.1.41**
Show that $\mathcal{M}^*$ is a matroid (*i.e.* show that $\{E \setminus B \ ; \ B \in \mathcal{B}_{\mathcal{M}}\}$ is a system of bases).

Theorem 1.1.42 *Duality is an involution on matroids, i.e. :* $(\mathcal{M}^*)^* = \mathcal{M}$.

⋆ **Exercise 1.1.43**
Prove this theorem.

⋆ **Exercise 1.1.44**
Show that: $\mathrm{rk}_{\mathcal{M}}(E) + \mathrm{rk}_{\mathcal{M}^*}(E) = |E|$

Show that for all $X \subseteq E$, we have $\mathrm{rk}_{\mathcal{M}^*}(X) + \mathrm{rk}_{\mathcal{M}}(E) = |X| + \mathrm{rk}_{\mathcal{M}}(E \setminus X)$.

⋆ **Exercise 1.1.45 — [Dual graphical matroid is a planar (in)variants]**
Recall that, for a planar graph drawn in the plane as a plane embedding $\overline{G}$, its plane-dual is the multi-graph obtained by setting one vertex in each face of $\overline{G}$, and an edge between two face of $\overline{G}$ seperated by and edge of $\overline{G}$.

Let G be a planar (multi-)graph together with a plane embedding, and G^* its plane-dual. Show that $\mathcal{M}_G^* = \mathcal{M}_{G^*}$.

Consider the graph G on 5 vertices a, b, c, a', b' and 5 edges ab, bc, ac, aa' and bb'. Find two different planar embeddings of G whose plane-dual are not isomorphic.

Using matroid language, show that the number of loops of the plane-dual is independent of the plane embedding of G. Imagine another non-trivial invariant.

⋆ **Exercise 1.1.46 — [Dual linear matroids]**
Show that the dual matroid of a linear matroid is a linear matroid (by explicitly constructing a collection of vectors).

(Semi-)open problem 1.1.47 *Is the dual of an algebraic matroid also an algebraic matroid?*

APPENDICE 1.B

Examples, and hierarchy of representability

⋆ Exercise 1.B.1 — [Uniform matroid]
The *uniform matroid* $\mathcal{U}_{n,k}$ of rank k is given by its system of bases $\binom{[n]}{k}$.
Write the independence system, the circuits, the lattice of flats, etc, of the uniform matroid $\mathcal{U}_{n,k}$.

⋆ Exercise 1.B.2 — [Non-Pappus matroid]
Consider the Pappus point configuration given 9 points in the plane satisfying: three points A, B, C aligned; three points a, b, c aligned; the point X is the intersection of the segments Ab and Ba; the point Y is the intersection of the segments Ac and Ca; and the point Z is the intersection of the segments Bc and Cb.
Draw the configuration, and prove (using a clever internet search on Pappus theorem) that X, Y, Z are aligned.
Let $\mathcal{M}$ be the matroid on ground set $(A, B, C, a, b, c, X, Y, Z)$ with circuits ABC, abc, AXb, AYc, BXa, BZc, CXa and CZb (but not XYZ). Deduce that the matroid $\mathcal{M}$ is not linear (nor graphical, nor transversal).

⋆ Exercise 1.B.3 — [Fano plane]
The *Fano plane* is the matroid $\mathcal{F}_7$ on 7 points obtained as follows: draw a triangle A, B, C and its 3 bissectors. These bissectors intersect together at a point G (the center of the inscribed circle), and intersect the sides of the triangle ABC at points D, E and F. The circuits of matroid $\mathcal{F}_7$ are all the tripletsof aligned points on this figure, together with the circuit DEF.
Show that $\mathcal{F}_7$ is realizable as a linear matroid over $\mathbb{F}_2$ using the all vectors in $\mathbb{F}_2^3$ except $(0, 0, 0)$.
Prove that $\mathcal{F}_7$ is not realizable over $\mathbb{Q}$.

Theorem 1.B.4 *For a matroid $\mathcal{M}$:*
Graphical ⇒ Linear ⇒ Algebraic
Transversal ⇒ Linear

⋆ Exercise 1.B.5 — [Graphical ⇒ Linear]
Let $G = (N, A)$ ("N" for nodes and "A" for arcs) be a graph an $\mathcal{M}(G)$ its associated graphical matroid. Let $\mathbb{F}^N$ be the vector space with canonical basis $(\boldsymbol{e}_u \ ; \ u \in N)$. Let $\boldsymbol{V} = (\boldsymbol{e}_u - \boldsymbol{e}_v \ ; \ uv \in A)$ be a vector configuration.
Show that $\mathcal{M}(G)$ is the linear matroid of the vector configuration $\boldsymbol{V}$ formed by one vector for each arc $uv \in A$:

$$\boldsymbol{e}_u - \boldsymbol{e}_v$$

⋆ Exercise 1.B.6 — [Transversal ⇒ Linear]
Let $G = (E \sqcup F, A)$ be a bipartite graph, and $\mathcal{M}(G)$ its associated transversal matroid. For each arc $ef \in A$, let X_{ef} be a transcendental number on $\mathbb{F}$ (*i.e.* a symbol with no algebraic relation with coefficient in $\mathbb{F}$). Let $\mathbb{F}^F$ be the vector space with canonical basis $(\boldsymbol{e}_f \ ; \ f \in F)$.
Show that $\mathcal{M}(G)$ is the linear matroid of the vector configuration $\boldsymbol{V}$ formed by one vector for each $e \in E$:

$$\sum_{f \in F} \sum_{ef \in A} X_{ef} \, \boldsymbol{e}_f$$

⋆ Exercise 1.B.7 — [Linear ⇒ Algebraic]
Let $\boldsymbol{V} = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) \subseteq \mathbb{F}^d$ be a vector configuration, and $\mathcal{M}(\boldsymbol{V})$ its associated linear matroid. In the field of rational functions $\mathbb{F}(X_1, \ldots, X_d)$, define for each $\boldsymbol{v}_i \in \boldsymbol{V}$, a rational function $f_i = \sum_{j=1}^{d} v_{i,j} \, X_j$.
Show that $\mathcal{M}(\boldsymbol{V})$ is the algebraic matroid of the collection of elements $\mathcal{V} = (f_1, \ldots, f_n)$.

Section 1.3

Vectors vs co-vectors, orientability

We focus on linear matroids, please translate everything to graphical, transversal and algebraic matroid each time.

♢ Construction 1.3.1 — [Vectors & circuits, co-vectors & co-circuits]
Fix a vector configuration $\boldsymbol{V} = (\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$.

Vectors: Consider a linear relation $\sum_{i=1}^n \lambda_i \boldsymbol{v}_i = \boldsymbol{0}$. The set $X = \{i \in [n] \; ; \; \lambda_i \neq 0\}$ is a *vector* of $\mathcal{M}_{\boldsymbol{V}}$. Conversely, the vectorial system of $\mathcal{M}_{\boldsymbol{V}}$ is the collection of all sets X, such that there exists $\sum_{i\in X} \lambda_i \boldsymbol{v}_i = \boldsymbol{0}$ with $\lambda_i \neq 0$ for all $i \in X$.
Circuits: The *circuits* of $\mathcal{M}_{\boldsymbol{V}}$ are its minimal vectors (*i.e.* the vectors $X \subseteq [n]$ such that if $Y \subseteq X$ is a vector, then $Y = X$). Circuits correspond to minimal linear dependencies.

Co-vectors: Consider any linear hyperplane (or equivalently linear function) $H_{\boldsymbol{c}} = \{\boldsymbol{x} \in \mathbb{R}^d \; ; \; \langle \boldsymbol{x}, \boldsymbol{c}\rangle = 0\}$, and let $X(\boldsymbol{c}) = \{i \in [n] \; ; \; \boldsymbol{v}_i \in H_{\boldsymbol{c}}\} = \{i \in [n] \; ; \; \langle \boldsymbol{v}_i, \boldsymbol{c}\rangle = 0\}$. A *co-vector* of the linear matroid $\mathcal{M}_{\boldsymbol{V}}$ is a subset $X \subseteq [n]$ such that there exists $\boldsymbol{c} \in \mathbb{R}^d$ with $X = X(\boldsymbol{c})$.
Co-circuits: The *co-circuits* of $\mathcal{M}_{\boldsymbol{V}}$ are the maximal co-vectors. Co-circuits, correspond to hyperplanes spanned by the vectors of $\boldsymbol{V}$ (becareful, if an hyperplane H is spanned by, say, the vectors $\boldsymbol{v}_{i_1}, \dots, \boldsymbol{v}_{i_{d+1}}$, then the corresponding co-circuit consist in **all** the indices i such that $\boldsymbol{v}_i \in H$, not only $i_1, \dots, i_{d+1}$).

⋆ Exercise 1.3.2
What about point configurations $\boldsymbol{A} = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)$ (*i.e.* linear → affine; linear combination → convex combination).

⋆ Exercise 1.3.3
What are the vectors, circuits, co-vectors and co-circuits of a graphical matroid?

⋆ Exercise 1.3.4
Show that C is a co-circuit of $\mathcal{M}$ if and only if $E \setminus C$ is a flat of $\mathcal{M}^*$.

Definition 1.3.5 — [Sign vector]
A sign vector is vector in $\{-1, 0, +1\}^n$, usually denoted like $\sigma = (+, 0, -, +, -, 0)$.
Alternatively, we can denote a sign vector as a triplet $X = (X_-, X_0, X_+)$ where $X_s = \{\sigma_i = s\}$.
The opposite of a sign vector is $-\sigma$ with $(-\sigma)_i = -(\sigma_i)$, or equivalently $-X = (X_+, X_0, X_-)$.

♢ Construction 1.3.6 — [Orienting everything]
Fix a vector configuration $\boldsymbol{V} = (\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$.

Oriented vectors: Consider a linear relation $\sum_{i=1}^n \lambda_i \boldsymbol{v}_i = \boldsymbol{0}$. The sign vector $X_+ = \{i \in [n] \; ; \; \lambda_i > 0\}$ and $X_- = \{i \in [n] \; ; \; \lambda_i < 0\}$ is an *oriented vector* for the configuration $\boldsymbol{V}$. Conversely, the oriented vectorial system for $\boldsymbol{V}$ is the collection of all such possible sign vectors.
Oriented circuits: The *oriented circuits* for $\boldsymbol{V}$ are its minimal oriented vectors (*i.e.* the oriented versions of the circuits of $\mathcal{M}_{\boldsymbol{V}}$). Oriented circuits correspond to minimal linear dependencies.

Oriented co-vectors: Consider any linear (open) half-space (or equivalently linear function) $H_{\boldsymbol{c}}^+ = \{\boldsymbol{x} \in \mathbb{R}^d \; ; \; \langle \boldsymbol{x}, \boldsymbol{c}\rangle > 0\}$, and let $X_+(\boldsymbol{c}) = \{i \in [n] \; ; \; \boldsymbol{v}_i \in H_{\boldsymbol{c}}^+\} = \{i \in [n] \; ; \; \langle \boldsymbol{v}_i, \boldsymbol{c}\rangle \geq 0\}$, and conversely for $X_-(\boldsymbol{c})$ and $X_0(\boldsymbol{c})$. An *oriented co-vector* for $\boldsymbol{V}$ is a signed vector X such that there exists $\boldsymbol{c} \in \mathbb{R}^d$ with $X = X(\boldsymbol{c})$.
Oriented co-circuits: The *oriented co-circuits* for $\boldsymbol{V}$ are the maximal oriented co-vectors. Oriented co-circuits, correspond to hyperplanes spanned by the vectors of $\boldsymbol{V}$.

⋆ Exercise 1.3.7
What about point configurations $\boldsymbol{A} = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)$ (*i.e.* linear → affine; linear combination → convex combination).

⋆ Exercise 1.3.8 — [Oriented circuits axioms]
For a directed graph G, with associate to each undirected cycle C with edges $(u_1u_2, u_2u_3, \dots, u_ru_1)$, a sign vector σ, called its *oriented circuit*, where $X_+ = \{i \in [r] \; ; \; u_i \to u_{i+1} \in G\}$ and $X_- = \{i \in [r] \; ; \; u_i \leftarrow u_{i+1}\}$ and $X_0 = \{e \in E \; ; \; e \notin C\}$.
Show that the system of oriented circuits $\mathcal{C}$ satisfies:

C1. $\boldsymbol{0} \notin \mathcal{C}$

C2. If $\sigma \in \mathcal{C}$, then $-\sigma \in \mathcal{C}$

C3. If $X, X' \in \mathcal{C}$ with $X_0 \subseteq X'_0$, then $X = X'$ or $X = -X'$

C4. (Strong circuit elimination) For all $X, X' \in \mathcal{C}$ with $X \neq -X'$, and all $e \in X_+ \cap X'_-$, then there exists $Y \in \mathcal{C}$ with $Y_- \subseteq (X_- \cup X'_-) \setminus \{e\}$ and $Y_+ = (X_+ \cup X'_+) \subseteq \setminus \{e\}$

Definition 1.3.9 — [System of oriented circuits]
A system of oriented circuits is a collection of sign vectors satisfying the above axioms (C1), (C2), (C3) and (C4).

Section 1.4

Basic notions of oriented matroids

Matroid world	Oriented matroid world
Graph	Directed graph
Point/vector alignments	Sign of determinants
Depdencies	Sign vectors
Linear dependencies between numbers	Sign of the coefficients

Definition 1.4.1 — [Oriented matroid]
An oriented matroid is a couple $(E, \mathcal{X})$ where E is the ground set (usually $E = [n]$), and $\mathcal{X}$ is a system of tuples of set or a collection of sign vectors (or anything) from which a system of oriented circuit can uniquely be retrieved.

The underlying matroid of an oriented matroid $\mathcal{M}$, denoted $\underline{\mathcal{M}}$ is the matroid whose co-circuits are the 0-sets of the oriented co-circuits of $\mathcal{M}$.

An oriented matroid is a linear/affine/graphical/transversal/algebraic oriented matroid if the underlying matroid is.

⋆ Exercise 1.4.2
Show that the circuits of the underlying matroid of a linear oriented matroid are the non-0-sets of its oriented circuits.

⋆ Exercise 1.4.3
Construct the notion of duality for oriented matroids.

What is the dual of a grahical oriented matroid?

Definition 1.4.4 — [Chirotope]
a chirotope of rank r on a ground set E is a function $\chi : E^r \to \{-, 0, +\}$ satisfying the three axioms:

B1. χ is not identically 0

B2. For any permutation σ of E, $\chi(x_{\sigma(1)}, \dots, x_{\sigma(r)}) = \varepsilon(\sigma)\, \chi(x_1, \dots, x_r)$, where $\varepsilon(\sigma)$ is the signature of σ

B3. For any $\boldsymbol{x}, \boldsymbol{y} \in E^r$, if for all $i \in [r]$ we have $\chi(y_i, x_2, \dots, x_r)\, \chi(y_1, \dots, y_{i-1}, x_i, y_{i+1}, \dots, y_r) \geq 0$, then $\chi(\boldsymbol{x})\chi(\boldsymbol{y}) \geq 0$

⋆ Exercise 1.4.5
For a vector configuration $\boldsymbol{V} = (\boldsymbol{v}_1, \dots, \boldsymbol{v}_n) \in \mathbb{R}^d$, show that $\chi(i_1, \dots, i_{d+1}) = \operatorname{sign}\det\big(\boldsymbol{v}_{i_1}, \dots, \boldsymbol{v}_{i_{d+1}}\big)$ is a chirotope of rank $d + 1$.

Crypto-morphism 1.4.6 — [Lawrence '82]
There is a bijection between systems of oriented circuits and chiroptopes.

⋆ Exercise 1.4.7
How can the underlying matroid of $\mathcal{M}$ be deduced from its chiroptope?

⋆ Exercise 1.4.8
Describe what is a graphical oriented matroid.

What is the chiroptope of a graphical oriented matroid?

Definition 1.4.9 — [Hyperplane arrangement]
An (linear) hyperplane in $\mathbb{R}^d$ is a convex set of $\mathbb{R}^d$ of the form $H_{\boldsymbol{a}} = \{\boldsymbol{x} \in \mathbb{R}^d \ ; \ \langle \boldsymbol{x}, \boldsymbol{a} \rangle = 0\}$ for some $\boldsymbol{a} \in \mathbb{R}^d$.

An affine hyperplane in $\mathbb{R}^d$ is a convex set of $\mathbb{R}^d$ of the form $H_{\boldsymbol{a},b} = \{\boldsymbol{x} \in \mathbb{R}^d \ ; \ \langle \boldsymbol{x}, \boldsymbol{a} \rangle = b\}$ for some $\boldsymbol{a} \in \mathbb{R}^d$ and $b \in R$.

An (affine) hyperplane arrangement is a collection of finitely many (affine) hyperplanes $\boldsymbol{H} = (H_1, \dots, H_n)$.

Definition 1.4.10 — [Half-space, oriented hyperplane arrangement]
A (closed) half-space is $H^+_{\boldsymbol{a},b} = \{\boldsymbol{x} \in \mathbb{R}^d \ ; \ \langle \boldsymbol{x}, \boldsymbol{a} \rangle \geq b\}$ for some $\boldsymbol{a} \in \mathbb{R}^d$ and $b \in \mathbb{R}$.

An oriented hyperplane arrangement is a collection of finitely many half-spaces $\mathcal{H} = (H^+_1, \dots, H^+_r)$.

♢ Construction 1.4.11
Let $\boldsymbol{H} = (H_1, \dots, H_r)$ be an hyperplane arrangement, and $\mathcal{H} = (H^+_1, \dots, H^+_n)$ an oriented hyperplane arrangement.

For each vector $\boldsymbol{x} \in \mathbb{R}^d$, the corresponding co-vector is the sub-set of $[n]$ defined as $S = \{i \in [n] \ ; \ \boldsymbol{x} \in H_i\}$; the corresponding oriented co-vector is the sign vector $(s_1, \dots, s_n)$ with $s_i = \begin{cases} 0 & \text{if } \boldsymbol{x} \in H_i \\ + & \text{if } \boldsymbol{x} \in H^+_i \smallsetminus H_i \\ - & \text{else} \end{cases}$.

A tope is a oriented co-vector with no 0.

⋆ Exercise 1.4.12 — [Braid arrangment]
The braid arrangement is the collection of hyperplanes $\boldsymbol{H}$ whose half-spaces are $\{\boldsymbol{x} \in \mathbb{R}^d \ ; \ x_i \leq x_j\}$ for all $i < j$.

What are the topes and the oriented co-vectors of the braid arrangement?

⋆ **Exercise 1.4.13**
What are the oriented circuits for a hyperplane arrangement?

Definition 1.4.14 — [Arrangement of spheres, pseudo-spheres]
A arrangement of sphere is the intersection of an (linear) arrangement of hyperplanes in $\mathbb{R}^d$ with the unit sphere $\mathbb{S}^d = \{\boldsymbol{x} \in \mathbb{R}^d \ ; \ \|x\| = 1\}$.

A pseudo-sphere is the image of $\mathbb{S}^{d-1}$ under[1] an homeomorphism $\mathbb{S}^{d-1} \to \mathbb{S}^d$. By (a nice extension of) Jordan's theorem, the complement of a pseudo-sphere S, namely $\mathbb{S}^d \setminus S$ has two connected components. An oriented pseudo-sphere is a pseudo-sphere together with a choice of an orientation, that is to say a positive S^+ and a negative S^- connected component, called the sides of the pseudo-sphere.

An arrangement of pseudo-spheres is a collection of pseud-spheres $(S_1, \dots, S_n)$ such that:

AS1. Every non-empty intersection of the S_i is (homeomorphic to) a sphere in some dimension.

AS2. For every non-empty intersection $S_A := \bigcap_{i \in A} S_i$, and every j such that $S_A \not\subseteq S_j$, the intersection $S_A \cap S_j$ is a pseudo-sphere in S_A with positive side $S_A \cap S_j^+$ and negative side $S_A \cap S_j^-$.

[1] Loosely speaking, it is a "wiggly" sphere of co-dimension 1, embedded onto the unit sphere.

Theorem 1.4.15 — [Falkman–Lawrence '78, Topological representation theorem]
For all oriented matroid $\mathcal{M}$, there exists an arrangement of pseudo-spheres whose oriented matroid is $\mathcal{M}$. Moreover, two arrangement of pseudo-spheres have the same oriented matroid if and only if they are equal up to homeomorphism.

Section 1.5

Minors, Tutte polynomial

Definition 1.5.1 — [Loop, co-loop]
For a matroid $\mathcal{M}$, a loop is an element of the ground set that is not in any independent set, and a co-loop is an element of the ground set that is a all bases.
The loops and co-loops of an oriented matroid are the loops and co-loops of its underlying matroid.

⋆ Exercise 1.5.2
Describe the loops and co-loops of linear/affine/graphical/transversal/algebraic matroid.

⋆ Exercise 1.5.3
Describe the loops and co-loops of an arrangement of hyperplanes/of pseudo-spheres.

Definition 1.5.4 — [Deletion, Contraction, Minors]
Let $\mathcal{M}$ be a matroid on the ground set E.
For $e \in E$ (usually not a co-loop), the deletion is defined as the matroid $\mathcal{M} \smallsetminus e$ on the ground set $E \smallsetminus \{e\}$ whose independence system is $\mathcal{I}_{\mathcal{M}\smallsetminus e} := \{S \ ; \ S \in \mathcal{I}_{\mathcal{M}} \text{ if } e \notin S\}$.
For $e \in E$ (usually not a loop), the contraction is defined as the matroid $\mathcal{M}/e$ on the ground set $E \smallsetminus \{e\}$ whose independence system is $\mathcal{I}_{\mathcal{M}/e} := \{S \smallsetminus \{e\} \ ; \ S \in \mathcal{I}_{\mathcal{M}}\}$.
A matroid $\mathcal{N}$ is a minor of a matroid $\mathcal{M}$ if $\mathcal{N}$ can be obtained from $\mathcal{M}$ by performing a sequence of deletion and contractions.

⋆ Exercise 1.5.5
Prove that deletion and contraction of matroids yield matroids.

⋆ Exercise 1.5.6
What are the bases of the deletion and contraction of a matroid.

⋆ Exercise 1.5.7
Describe deletion and contractions of linear/affine/graphical/transversal/algebraic matroid.

⋆ Exercise 1.5.8 — [Minors of oriented matroids]
Describe the notions of deletion and contraction for oriented matroids.
What does it do in an arrangement of hyperplanes/of pseudo-spheres?

⋆ Exercise 1.5.9 — [Graphical arrangement]
For a graph G (undirected, without loops nor bridges) on vertex set V, the corresponding *graphical arrangement* is the (linear) hyperplane arrangement $\mathcal{H}_G$ formed by the hyperplanes $H_{uv} = \{\boldsymbol{x} \in \mathbb{R}^V \ ; \ x_u = x_v\}$ for each edge uv of the graph G.
The braid arrangement is the graphical arrangement of a graph: which graph?
Show that the topes of the graphical arrangement correspond to the the acyclic orientations of G.
Show that the circuits of the graphical arrangement correspond to strong orientations of G. Deduce Robbins theorem: an undirected graph admits a strong orientation if and only if it is 2-connected and loop-less.

Definition 1.5.10 — [Acyclic orientation]
An acyclic orientation of a matroid M is an orientation matroid $\mathcal{M}$ such that $M = \underline{\mathcal{M}}$ and the all-positive sign vector $(+, +, \dots, +)$ is a tope of $\mathcal{M}$.

One aim of the Tutte polynomial is to count the number of acyclic orientations of a matroid. In the meanwhile, we will recover a notion that is stronger than the f-vector (actually f-polynomial) of a simplicial complex, but has the same flavor.

Definition 1.5.11 — [Tutte polynomial]
For a matroid or an oriented matroid $\mathcal{M}$ on the ground set E, the following polynomial is called the Tutte polynomial (a.k.a. Whitney corank-nullity polynomial, historically Crapo's polynomial):

$$T_{\mathcal{M}}(x+1, y+1) = \sum_{A \subseteq E} x^{\mathrm{rk}_{\mathcal{M}}(E) - \mathrm{rk}_{\mathcal{M}}(A)} \, y^{|A| - \mathrm{rk}_{\mathcal{M}}(A)}$$

Theorem 1.5.12 *The Tutte polynomial $T_{\mathcal{M}}$ of a matroid $\mathcal{M}$ on the ground set E satisfies:*

T1. If $e \in E$ is neither a loop nor a co-loop, then: $T_{\mathcal{M}}(x, y) = T_{\mathcal{M}\smallsetminus e}(x, y) + T_{\mathcal{M}/e}(x, y)$.

T2. If $e \in E$ loop, then: $T_{\mathcal{M}}(x, y) = xT_{\mathcal{M}/e}(x, y)$; if $e \in E$ a co-loop: $T_{\mathcal{M}}(x, y) = yT_{\mathcal{M}\smallsetminus e}(x, y)$.

T3. If $E = \{e\}$, then $T_{\mathcal{M}} = \begin{cases} x & \text{if } e \text{ is a coloop} \\ y & \text{if } e \text{ is a loop} \end{cases}$.

T4. $T_{\mathcal{M}^*}(x, y) = T_{\mathcal{M}}(y, x)$.

T5. $T_{\mathcal{M}_1 \oplus \mathcal{M}_2}(x, y) = T_{\mathcal{M}_1}(x, y) \cdot T_{\mathcal{M}_2}(x, y)$

Furthermore, $T_{\mathcal{M}}$ is characterized by the properties (T1), (T2) and (T3); or by (T1), (T3) and (T5).

⋆ Exercise 1.5.13

Prove the above theorem.

Definition 1.5.14 — [Contraction-deletion invariants]

Two matroids $\mathcal{M}$ (on ground set E) and $\mathcal{N}$ (on ground set F) if there exists a bijection $\sigma : E \to F$ such that $\mathcal{I}_{\mathcal{N}} = \{\{\sigma(x)\ ;\ x \in S\}\ ;\ S \in \mathcal{I}_{\mathcal{M}}\}$.

An invariant Ψ on matroids is a map from the set of all matroids towards, say, a commutative ring R, such that two isomorphic matroids get the same image by Ψ.

An invariant is a contraction-deletion invariant if there exists $a, b, c, d \in R$ such that the following hold:

- $\Psi(\mathcal{M}_1 \oplus \mathcal{M}_2) = \Psi(\mathcal{M}_1)\Psi(\mathcal{M}_2)$
- *If e is neither a loop nor a co-loop, then* $\Psi(\mathcal{M}) = a\Psi(\mathcal{M} \setminus e) + b\Psi(\mathcal{M}/e)$
- *If $E = \{e\}$, then* $\Psi(\mathcal{M}) = \begin{cases} c & \text{if } e \text{ is a coloop} \\ d & \text{if } e \text{ is a loop} \end{cases}$

Theorem 1.5.15 — [Contraction-deletion invariants "are" Tutte polynomials]

The invariant Ψ is a constraction-deletion invariant if and only if:

$$\Psi(\mathcal{M}) = a^{|E| - \mathrm{rk}_{\mathcal{M}}(E)}\, b^{\mathrm{rk}_{\mathcal{M}}(E)}\, T_{\mathcal{M}}\left(\frac{c}{b}, \frac{d}{a}\right)$$

⋆ Exercise 1.5.16

Show that $T_{\mathcal{M}}(1, 1)$ is the number of bases of $\mathcal{M}$.

Show that $T_{\mathcal{M}}(2, 2) = E^{|E|}$ where E is the ground set of $\mathcal{M}$.

Show that $T_{\mathcal{M}}(2, 1)$ is the number of independent sets of $\mathcal{M}$.

Show that $T_{\mathcal{M}}(1, 2)$ is the number ofsets which contains a basis of $\mathcal{M}$.

⋆ Exercise 1.5.17 — [Chromatic polynomial of a graph]

For a graph G on vertex set V, a *k-coloring* is a function $c : V \to [k]$ such that $c(u) \neq c(v)$ if uv is an edge of G. The *chromatique polynomial* χ_G of G is given by $\chi_G(k) =$ number of k-coloring of G.

Show that $G \mapsto \chi_G$ is a contraction-deletion invariant of graphical matroids, and deduce that (with $c(G)$ is the number of connected components of G):

$$\chi_G(k) = (-1)^{|V| - c(G)} k^{c(G)} T_{\mathcal{M}_G}(1 - k,\, 0)$$

⋆ Exercise 1.5.18 — [Flow polynomial]

A *nowhere-zero k-flow* on an oriented graph G is a map $f : E \to [k-1]$ such that for each vertex v of G, $\sum_{u \to v} f(u \to v) \equiv \sum_{v \to w} f(v \to w)\ (mod k)$. The *flow polynomial* C_G of G is given by $C_G(k) =$ number of nowhere-zero k-flows of G.

Show that $G \mapsto C_G$ is a contraction-deletion invariant of graphical matroids, and deduce that (with $c(G)$ is the number of connected components of G):

$$C_G(k) = (-1)^{|E| - |E| + c(G)} T_{\mathcal{M}_G}(0,\, 1 - k)$$

⋆ Exercise 1.5.19 — [Acyclic orientations]

A *k-weak topological order*[1] on a graph G is an acyclic orientation $\mathcal{O}$ of G together with a map $\sigma : V \to [k]$ which is compatible with $\mathcal{O}$ (*i.e.* if $u \to v \in \mathcal{O}$, then $\sigma(u) \leq \sigma(v)$). The *Stanley polynomial* $\overline{\chi}_G$ of a graph G is given by $\overline{\chi}_G =$ number of k-weak topological order of G.

Show that $G \mapsto \overline{\chi}_G$ is a contraction-deletion invariant of graphical matroids, and deduce that (with $c(G)$ is the number of connected components of G):

$$\chi_G(k) = (-1)^{|V|} \chi_G(-k) = (-1)^{c(G)} k^{c(G)} T_{\mathcal{M}_G}(1 + k,\, 0)$$

Conclude that the number of acyclic orientations of G is given by (up to sign) $\chi_G(-1)$.

⋆ Exercise 1.5.20

Conclude your understanding of the Tutte polynomial by agreeing with the following sentences:

"The number of topes of an hyperplane arrangement is given by the evaluation of its Tutte polynomial at $(2, 0)$."

"The number of acyclic (re)orientation of a (oriented) matroid is given by the evaluation of its Tutte polynomial at $(2, 0)$."

[1] A *topological order* on a directed graph D is a map $\sigma : V \to [|V|]$ such that if $u \to v$ in D, then $\sigma(u) < \sigma(v)$.

Section 1.6

To go further beyond

Have a look at (the Wikipedia pages of):

- Tutte polynomial for graphs, Tutte plane
- Gale transform
- Matroid polytope
- Localization of tope graphs of oriented matroids
- Lexicographic extensions of oriented matroids

Part I

Index et Bibliography

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